# Problems 1

**1.1** Substituting the operators $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$ and $E = i\hbar\partial/\partial t$ into the mass-energy relation $E^2 = p^2c^2 + M^2c^4$ and allowing the operators to act on the function $\phi(\mathbf{r}, t)$, leads immediately to the Klein-Gordon equation. To verify that the Yukawa potential $V(r)$ is a static solution of the equation, set $V(r) = \phi(\mathbf{r})$, where $r = |\mathbf{r}|$, and use

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

together with the expression for the range, $R = \hbar/Mc$.

**1.2** Using the relations (1.11), gives

$$\hat{P}Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin(\pi - \theta)e^{i(\pi+\phi)} = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi} = -Y_1^1,$$

and hence $Y_1^1$ is an eigenfunction of parity with eigenvalue $-1$.

**1.3** Because the initial state is at rest, it has $L = 0$ and thus its parity is $P_i = P_p P_{\bar{p}}(-1)^L = -1$, where we have used the fact that the fermion-antifermion pair has overall negative intrinsic parity. In the final state, the neutral pions are identical bosons and so their wavefunction must be totally symmetric under their interchange. This implies even orbital angular momentum $L'$ between them and hence $P_f = P_\pi^2(-1)^{L'} = 1 \neq P_i$. The reaction does not conserve parity and is thus forbidden as a strong interaction.

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**1.4** Since $\hat{C}^2 = 1$, we must have

$$\hat{C}^2 \left| b, \psi_b \right\rangle = C_b \hat{C} \left| \bar{b}, \psi_{\bar{b}} \right\rangle = \left| b, \psi_b \right\rangle ,$$

implying that

$$\hat{C} \left| \bar{b}, \psi_{\bar{b}} \right\rangle = C_{\bar{b}} \left| b, \psi_b \right\rangle$$

with $C_b C_{\bar{b}} = 1$ independent of $C_b$. The result follows because an eigenstate of $\hat{C}$ must contain only particle-antiparticle pairs $b\bar{b}$, leading to the intrinsic parity factor $C_b C_{\bar{b}} = 1$, independent of $C_b$.

**1.5** Under $C$-conjugation, $\hat{C}\mathbf{A}(\mathbf{r}, t) = C_\gamma \mathbf{A}(\mathbf{r}, t)$, where $C_\gamma$ is the $C$-parity of the photon. But we also have $\hat{C}\mathbf{E}(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t)$, because electric charges change sign. Thus, if the equation $\mathbf{E} = -\partial\mathbf{A}\big/\partial t$ is invariant under charge conjugation, $C_\gamma = -1$.

**1.6** The electric dipole moment is the expectation value of the operator $\mathbf{d} = \sum_i q_i \mathbf{r}_i$ and is a vector. Thus $\mathbf{d} \propto \mathbf{J}$, the angular momentum, and since $\mathbf{J}$ changes sign under time reversal (classically $\mathbf{J} = \mathbf{r} \times \mathbf{p}$), so does $\mathbf{d}$, whereas the electric field $\mathbf{E}$ does not. Thus the interaction energy $\mathbf{d} \cdot \mathbf{E}$ can only be invariant under time reversal if $\mathbf{d} \equiv \mathbf{0}$.

**1.7** From Equations (1.73) and (1.74), we have

$$R = \frac{\mathrm{d}\sigma(pp \to \pi^+ d)/\mathrm{d}\Omega}{\mathrm{d}\sigma(\pi^+ d \to pp)/\mathrm{d}\Omega} = \frac{(2s_\pi + 1)(2s_d + 1)}{(2s_p + 1)^2} \frac{p_\pi^2}{p_p^2} \frac{\overline{|\mathcal{M}_{if}|^2}}{\overline{|\mathcal{M}_{fi}|^2}}.$$

By detailed balance, the two spin-averaged squared matrix elements are equal, and so, using $s_p = \frac{1}{2}$ and $s_d = 1$, gives the result.

**1.8** The parity of the deuteron is $P_d = P_p P_n (-1)^{L_{pn}}$. Since the deuteron is an S-wave bound state, $L_{pn} = 0$ and so, using $P_p = P_n = 1$, gives $P_d = 1$. The parity of the initial state is therefore

$$P_i = P_{\pi^-} P_d (-1)^{L_{\pi d}} = P_{\pi^-},$$

because the pion is at rest and so $L_{\pi d} = 0$. The parity of the final state is

$$P_f = P_n P_n (-1)^{L_{nn}} = (-1)^{L_{nn}}$$

and therefore $P_{\pi^-} = (-1)^{L_{nn}}$. To find $L_{nn}$ impose the Pauli principle condition that $\psi_{nn} = \psi_{\text{space}} \psi_{\text{spin}}$ must be antisymmetric. Examining the spin wavefunctions (1.17) shows that there are two possibilities for $\psi_{\text{spin}}$: either the symmetric $S = 1$ state or the $S = 0$ antisymmetric state. If $S = 0$, then $\psi_{\text{space}}$ would have to be symmetric, implying $L_{nn}$ would be even. But the total angular momentum would not then be conserved. Thus $S = 1$ is implied and $\psi_{\text{space}}$ is antisymmetric, i.e. $L_{nn} = 1, 3, \cdots$ The only way to combine $L_{nn}$ and $S$ to give $J = 1$ is with $L_{nn} = 1$ and hence $P_{\pi^-} = -1$.

**1.9** (a) $\bar{\nu}_e + e^+ \to \bar{\nu}_e + e^+$;

(b) $p + p \to p + p + \pi^0 + \pi^0$;

(c) $\bar{p} + n \to \pi^- + \pi^0 + \pi^0, \quad \pi^- + \pi^+ + \pi^-$.

**1.10** (a) $\nu_e + \nu_\mu \to \nu_e + \nu_\mu$

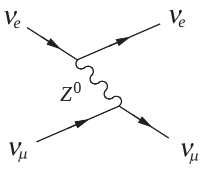


**Figure F.1**

(b) $e^+ + e^- \to e^+ + e^-$

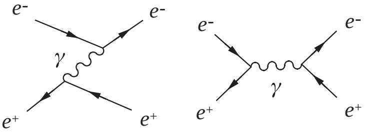


**Figure F.2**

(c) $\gamma + \gamma \to e^+ + e^-$

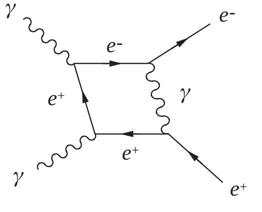


**Figure F.3**

**1.11** If an exchanged particle approaches to within a distance $d$ fm, this is equivalent to a momentum transfer $q = \hbar/d = (0.2/d)$ GeV/c. Thus, $q = 0.2$ GeV/c for $d = 1$ fm and $q = 200$ GeV/c for $d = 10^{-3}$ fm. The scattering amplitude is given by

$$f(q^2) = -g^2\hbar^2 \left(q^2 + m_x^2 c^2\right)^{-1},$$

where $m_x$ is the mass of the exchanged particle. Thus,

$$R(q^2) \equiv \frac{f_{\text{EM}}(q^2)}{f_{\text{Weak}}(q^2)} = \frac{q^2c^2 + m_W^2 c^4}{q^2c^2 + m_\gamma^2 c^4},$$

since $g_{\text{EM}} \approx g_{\text{Weak}}$. Using $m_\gamma = 0$ and $m_W = 80$ GeV/C$^2$, gives

$$R(0.2 \text{ GeV/c}) = 1.60 \times 10^5 \quad \text{and} \quad R(200 \text{ GeV/c}) = 1.16.$$

**1.12** Using spherical polar co-ordinates, we have

$$\mathbf{q} \cdot \mathbf{r} = qr \cos\theta \quad \text{and} \quad \mathrm{d}^3\mathbf{r} = r^2 \mathrm{d}r \ \mathrm{d}\cos\theta \ \mathrm{d}\phi,$$

where $q = |\mathbf{q}|$. Thus, from (1.47),

$$\mathcal{M}(q^2) = \frac{-g^2}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}r\, r^2 \frac{e^{-rR}}{r} \int_{-1}^{+1} \mathrm{d}\cos\theta \exp(iqr\cos\theta/\hbar) = \frac{-g^2\hbar^2}{q^2 + m^2c^2}.$$

**1.13** Let one of the beams (labelled by 1) refer to the 'beam' and let the other beam (labelled by 2) refer to the 'target'. Then in (1.62),

$$n_b = nN_1/2\pi RA \quad \text{and} \quad \upsilon_i = 2\pi R/T,$$

where $R$ is the radius of the circular path. Thus the flux is

$$J = n_b\upsilon_i = nN_1 f/A,$$

where $f$ is the frequency. Also $N = N_2$, so finally the luminosity is

$$L = JN = nN_1N_2 f/A.$$

**1.14** From (1.57c), $\sigma = WM_A/I(\rho t)N_A$. Since the scattering is isotropic, the total number of protons emitted per second from the target is

$$W = 20 \times (4\pi/2 \times 10^{-3}) = 1.25 \times 10^5 \text{ s}^{-1}.$$

$I$ can be calculated from the current, noting that the alpha particles carry two units of charge, and is $I = 3.13 \times 10^{10}$ s$^{-1}$. The density of the target is $\rho t = 10^{-32}$ kg fm$^{-2}$. Putting everything together gives $\sigma = 161$ mb.

**1.15** Write

$$\sigma = \frac{8\pi\alpha^2}{3m_e^2}\hbar^a c^b$$

and impose the dimensional condition $[\sigma] = [L]^2$. This gives $a = -b = 2$ and hence

$$\sigma = \frac{8\pi\alpha^2(\hbar c)^2}{3(m_ec^2)^2}.$$

Evaluating this gives $\sigma = 6.65 \times 10^{-29}$ m$^2 = 0.665$ mb.

## Problems 2

**2.1** From Equation (2.22), the accuracy $A$ is given by $A \propto \left(t\sqrt{N}\right)^{-1}$, where $t = T_{obs}$. For radioisotopes, $N \propto \exp(-\lambda t)$, with $\lambda = \ln 2/t_{1/2}$, where $t_{1/2}$ is the half-life. Thus,

$$A \propto \frac{1}{t}\exp\left(\frac{t\ln 2}{2t_{1/2}}\right),$$

which has a maximum at $t/t_{1/2} \approx 2.9$.

**2.2** From (2.32), the form factor is

$$F(q^2) = 3[\sin b(a) - b(a)\cos b(a)]b^{-3},$$

where $b = qa/\hbar$. To evaluate this we need to find $a$ and $q$. We can find the latter from Figure F.4.

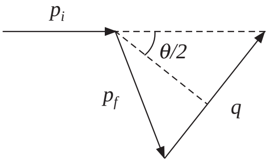


**Figure F.4**

Using $p = |\mathbf{p}_i| = |\mathbf{p}_f|$ and $q = |\mathbf{q}|$, gives

$$q = 2p\sin(\theta/2) = 57.5 \text{ MeV/c}.$$

Also, we know that $a = 1.21A^{1/3}$ fm and so for $A = 56$, $a = 4.63$ fm and $qa/\hbar = 1.35$ radians. Finally, using this in the expression for $F$, gives $F = 0.829$ and hence the reduction is $F^2 = 0.69$.

**2.3** Setting $q = |\mathbf{q}|$ in (2.37), we have

$$F(\mathbf{q}^2) = \frac{1}{Ze}\int f(\mathbf{r})\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{irq\cos\theta}{\hbar}\right)^n \mathrm{d}^3\mathbf{r}.$$

Using $\mathrm{d}^3\mathbf{r} = r^2\mathrm{d}\cos\theta\ \mathrm{d}\phi\mathrm{d}r$ and doing the $\phi$ integral, gives

$$\begin{aligned} F(\mathbf{q}^2) &= \frac{2\pi}{Ze}\iint f(r)r^2\left[1 + \frac{iqr\cos\theta}{\hbar} - \frac{q^2r^2\cos^2\theta}{2\hbar^2} + \ldots\right]\mathrm{d}r\ \mathrm{d}\cos\theta \\ &= \frac{4\pi}{Ze}\int_0^{\infty} f(r)r^2\mathrm{d}r - \frac{4\pi q^2}{6Ze\hbar^2}\int_0^{\infty} f(r)r^4\mathrm{d}r + \ldots. \end{aligned}$$

But from (2.28),

$$Ze = 4\pi\int_0^{\infty} f(r)r^2\mathrm{d}r$$

and from (2.36)

$$Ze\langle r^2\rangle = 4\pi\int_0^{\infty} f(r)r^4\mathrm{d}r,$$

so

$$F(\mathbf{q}^2) = 1 - \frac{\mathbf{q}^2}{6\hbar^2}\langle r^2\rangle + \cdots$$

**2.4** From (2.39),

$$\langle r^2\rangle = 6\hbar^2[1 - F(q^2)]/q^2,$$

where

$$q = 2(E/c)\sin(\theta/2) = 43.6 \text{ MeV/c}$$

Also, $F^2 = 0.65$ and so $\sqrt{\langle r^2 \rangle} = 4.88$ fm.

**2.5** The charge distribution is spherical, so the angular integrations in the general result (2.28) may be done, giving

$$F(\mathbf{q}^2) = \left[\int_0^\infty \rho(r)[\sin(qr/\hbar)/(qr/\hbar)]4\pi r^2 \mathrm{d}r\right]\left[\int_0^\infty \rho(r)4\pi r^2 \mathrm{d}r\right]^{-1}.$$

Substituting for $\rho(r)$, setting $x = r/a$ and using $\int_0^\infty x\exp(-x)\,dx = 1$, gives, after integrating by parts (twice),

$$F(\mathbf{q}^2) = \left(\frac{\hbar}{qa}\right)\int_0^\infty e^{-x}\sin\left(\frac{qax}{\hbar}\right)\mathrm{d}x = \frac{1}{\left(1 + q^2a^2/\hbar^2\right)}.$$

**2.6** In 1 g of the isotope there are initially

$$N_0 = (1 \text{ g}/208 \times 1.66 \times 10^{-24} \text{ g}) = 2.9 \times 10^{21} \text{ atoms}.$$

At time $t$ there are $N(t) = N_0 e^{-t/\tau}$ atoms, where $\tau$ is the mean life of the isotope. Provided $t \ll \tau$, the average decay rate is

$$\frac{N_0 - N(t)}{t} \approx \frac{N_0}{\tau} = \frac{75}{0.1 \times 24} \text{ hr}^{-1}.$$

Thus, $\tau = 2.4N_0/75$ hr $\approx 1.1 \times 10^{16}$ yr.

**2.7** The count rate is proportional to the number of $^{14}$C atoms present in the sample. If we assume that the abundance of $^{14}$C has not changed with time, and that the artifact was made from living material and is predominantly carbon, then at the time it was made ($t = 0$), 1 g would have contained $5 \times 10^{22}$ carbon atoms of which $N_0 = 6 \times 10^{10}$ would have been $^{14}$C. Thus the average count rate would have been $N_0/\tau = 13.8$ m$^{-1}$. At time $t$, the number of $^{14}$C atoms would be $N(t) = N_0\exp(-t/\tau)$ and

$$N(t)/N_0 = e^{-t/\tau} = 2.1/13.8,$$

from which $t = \tau \ln 6.57 = 1.56 \times 10^4$ yr. The artifact is approximately 16,000 years old.

**2.8** If the transition rate for $^{212}_{86}$Rn decay is $\omega_1$ and that for $^{208}_{84}$Po is $\omega_2$ and if the numbers of each of these atoms at time $t$ is $N_1(t)$ and $N_2(t)$, respectively, then the decays are governed by Equation (2.67), i.e.

$$N_2(t) = \omega_1 N_1(0)\left[\exp(-\omega_1 t) - \exp(-\omega_2 t)\right](\omega_2 - \omega_1)^{-1}.$$

The latter is a maximum when $\mathrm{d}N_2(t)/\mathrm{d}t = 0$, i.e. when

$$\omega_2\exp(-\omega_2 t) = \omega_1\exp(-\omega_1 t),$$

with

$$t_{max} = \ln(\omega_1/\omega_2)\,(\omega_1 - \omega_2)^{-1}\,.$$

Using

$$\omega_1 = 4.12 \times 10^{-2}\ \text{min}^{-1} \quad \text{and} \quad \omega_2 = 6.56 \times 10^{-7}\ \text{min}^{-1},$$

gives $t_{max} = 268$ min.

**2.9** The total decay rate of both modes of $^{138}_{57}\text{La}$ is

$$(1 + 0.5) \times (7.8 \times 10^2)\ \text{kg}^{-1}\ \text{s}^{-1} = 1.17 \times 10^3\ \text{kg}^{-1}\ \text{s}^{-1}.$$

Also, since this isotope is only 0.09 % of natural lanthanum, the number of $^{138}_{57}\text{La}$ atoms/kg is

$$N = (9 \times 10^{-4}) \times (1000/138.91) \times (6.022 \times 10^{23}) = 3.90 \times 10^{21}\ \text{kg}^{-1}.$$

The rate of decays is $-\mathrm{d}N/\mathrm{d}t = \omega\ N$, where $\omega$ is the transition rate and in terms of this, the mean lifetime $\tau = 1/\omega$. Thus,

$$\tau = \frac{N}{-\mathrm{d}N/\mathrm{d}t} = 3.33 \times 10^{18}\ \text{s} = 1.06 \times 10^{11}\ \text{yr}.$$

**2.10** The energy released is the increase in binding energy. Now from the SEMF,

$$BE(35, 87) = a_\upsilon(87) - a_s(87)^{2/3} - a_c\frac{(35)^2}{(87)^{1/3}} - a_a\frac{(87-70)^2}{348},$$

$$BE(57, 145) = a_\upsilon(145) - a_s(145)^{2/3} - a_c\frac{(57)^2}{(145)^{1/3}} - a_a\frac{(145-114)^2}{580},$$

$$BE(92, 235) = a_\upsilon(235) - a_s(235)^{2/3} - a_c\frac{(92)^2}{(235)^{1/3}} - a_a\frac{(235-184)^2}{940}.$$

The energy released is thus

$$\begin{aligned} E &= BE(35, 87) + BE(57, 145) - BE(92, 235) \\ &= -3\,a_\upsilon - 9.153\,a_s + 476.7a_c + 0.280\,a_a \end{aligned}$$

which, using the values given in (2.57), gives $E = 154$ MeV.

**2.11** The most stable nucleus for fixed $A$ has a $Z$-value given by i.e. $Z = \beta/2\gamma$, where from Equation (2.71),

$$\beta = a_a + (M_n - M_p - m_e) \quad \text{and} \quad \gamma = a_a/A + a_c/A^{1/3}.$$

Changing $\alpha$ would not change $a_a$, but would effect the Coulomb coefficient because $a_c$ is proportional to $\alpha$. For $A = 111$, using the value of $a_a$ from Equation (2.57) gives

$$\beta = 93.93\,\text{MeV/c}^2 \quad \text{and} \quad \gamma = 0.839 + 0.208\ a_c\,\text{MeV/c}^2.$$

For $Z = 47$, $a_c = 0.770$ MeV/c$^2$. This is a change of about 10 % from the value given in (2.57) and so $\alpha$ would have to change by the same percentage.

**2.12** In the rest frame of the $^{269}_{108}$Hs nucleus, $m_\alpha \upsilon_\alpha = m_{Sg}\upsilon_{Sg}$, where $\upsilon$ is the appropriate velocity. The ratio of the kinetic energies in the centre-of-mass is therefore $E_{Sg}/E_\alpha = m_\alpha/m_{Sg}$ and the total kinetic energy is

$$E_\alpha \left(1 + m_\alpha/m_{Sg}\right) = 9.369 \text{ MeV}.$$

Thus,

$$m_{Hs}c^2 = (m_{Sg} + m_\alpha)c^2 + 9.369 \text{ MeV} = 269.133 \text{ u}.$$

**2.13** If there are $N_0$ atoms of $^{238}_{94}$Pu at launch, then after $t$ years the activity of the source will be $\mathcal{A}(t) = N_0 \exp(-t/\tau)/\tau$, where $\tau$ is the lifetime. The instantaneous power is then

$$P(t) = \mathcal{A}(t) \times 0.05 \times 5.49 \times 1.602 \times 10^{-13} \text{ W} > 200\ W.$$

Substituting the value given for $\tau$, gives $N_0 = 1.88 \times 10^{25}$ and hence the weight of $^{238}_{94}$Pu at launch would have to be at least

$$\left(\frac{1.88 \times 10^{25}}{6.02 \times 10^{23}}\right)\left(\frac{238}{1000}\right) \text{kg} = 7.43 \text{ kg}.$$

**2.14** If there were $N_0$ atoms of each isotope at the formation of the planet ($t = 0$), then after time $t$ the numbers of atoms of $^{205}$Pb is

$$N_{205}(t) = N_0 \exp(-t/\tau_{205}), \quad \text{with} \quad N_{204}(t) = N_0,$$

so that

$$\frac{N_{205}(t)}{N_{204}(t)} = \exp\left(-\frac{t}{\tau_{205}}\right) = \frac{n_{205}}{n_{204}} = 2 \times 10^{-7}$$

and $t = -\tau_{205} \ln(2 \times 10^{-7}) = 2.4 \times 10^8$ y.

**2.15** We first calculate the mass difference between $\left(p + {}^{46}_{21}\text{Sc}\right)$ and $\left(n + {}^{46}_{22}\text{Ti}\right)$. Using the information given, we have

$$M(21, 46) - [M(22, 46) + m_e] = 2.37 \text{ MeV/c}^2$$

and

$$M_n - (M_p + m_e) = 0.78 \text{ MeV/c}^2$$

and hence

$$[M_p + M(21, 46)] - [M_n + M(22, 46)] = 1.59 \text{ MeV/c}^2.$$

We also need the mass differences

$$[M_\alpha + M(20, 43)] - [M_n + M(22, 46)] = 0.07 \text{ MeV/c}^2.$$

We can now draw the energy level diagram where the centre-of-mass energy of the resonance is (see Equation (2.8)) $2.76 \times (45/47) = 2.64$ MeV.

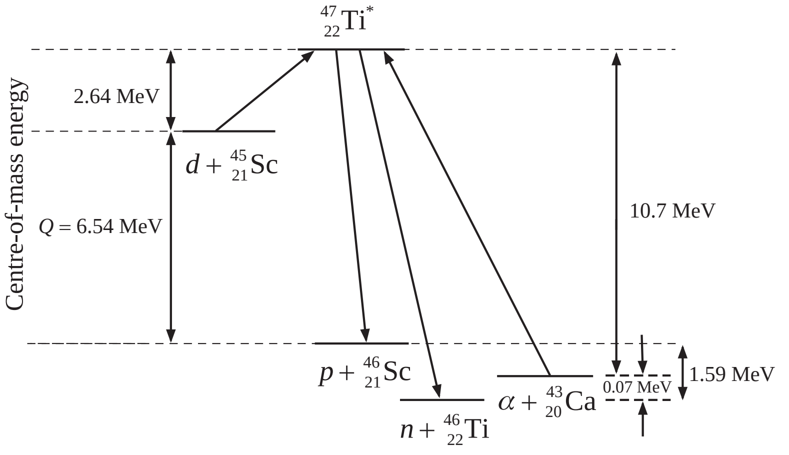


**Figure F.5**

Thus the resonance could be excited in the ${}^{43}_{20}$Ca$(\alpha, n){}^{46}_{22}$Ti reaction at an $\alpha$ particle laboratory energy of $10.7 \times (47/43) = 11.7$ MeV.

**2.16** We have $\mathrm{d}N(t)/\mathrm{d}t = P - \lambda N$, from which

$$Pe^{\lambda t} = e^{\lambda t}\left(\lambda N + \frac{\mathrm{d}N(t)}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(Ne^{\lambda t}\right)$$

Integrating and using the fact that $N = 0$ at $t = 0$ to determine the constant of integration, gives the required result.

**2.17** The number of $^{35}$Cl atoms in 1 g of the natural chloride is

$$N = 2 \times 0.758 \times N_A/\text{molecular weight} = 7.04 \times 10^{21}.$$

The activity

$$\mathcal{A}(t) = \lambda N = P\left(1 - e^{-\lambda t}\right) \approx P\lambda t, \ \text{since } \lambda t \ll 1.$$

So

$$t = \frac{\mathcal{A}(t)}{P\lambda} = \frac{\mathcal{A}(t)t_{1/2}}{\ln 2 \times \sigma \times F \times N}.$$

Substituting $\mathcal{A}(t) = 3 \times 10^5$ Bq and using the other constants given, yields $t = 1.55$ days.

**2.18** At very low energies we may assume the scattering has $l = 0$ and so in Equation (1.80) we have $j = \frac{1}{2}$, $s_n = \frac{1}{2}$ and $s_u = 0$. Thus,

$$\sigma_{max} = \frac{\pi\hbar^2}{q_n^2}\frac{(\Gamma_n\Gamma_n + \Gamma_n\Gamma_\gamma)}{\Gamma^2/4} = \frac{4\pi\hbar^2\Gamma_n}{q_n^2\Gamma},$$

Therefore, $\Gamma_n = q_n^2\Gamma\sigma_{max}/4\pi\hbar^2 = 0.35 \times 10^{-3}$ eV and $\Gamma_\gamma = \Gamma - \Gamma_n = 9.65 \times 10^{-3}$ eV.

## Problems 3

**3.1** (a) Forbidden: violates $L_\mu$ conservation, because $L_\mu(\nu_\mu) = 1$, but $L_\mu(\mu^+) = -1$.
(b) Forbidden: violates electric charge conservation, because $Q$ (left-hand side) $= 1$, but $Q$ (right-hand side) $= 0$
(c) Forbidden: violates baryon number conservation because $B$ (left-hand side) $= 1$, but $B$ (right-hand side) $= 0$
(d) Allowed as a weak interaction: conserves $L_\mu$, $B$, $Q$ etc., violates $S$. (It also has $\Delta S = \Delta Q$ for the hadrons – this is discussed in Section 6.5.2.)
(e) Allowed as a weak interaction: conserves $L_e$, $B$, $Q$ etc.
(f) Forbidden: violates $L_\mu$ and $L_\tau$

**3.2** A possible Feyman diagram is shown in Figure F.6.

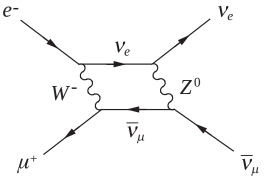


**Figure F.6**

**3.3** From (3.32) we have $L_0 = 4E(\hbar c)/\Delta m_{ij}^2 c^4$. Then if $L_0$ is expressed in km, $E$ in GeV and $\Delta m_{ij}^2$ in (eV/c$^2$)$^2$, we have

$$L_0 = \frac{4E \times (1.97 \times 10^{-13}) \times 10^{18}}{\Delta m_{ij}^2} = \frac{E}{1.27\Delta m_{ij}^2} \text{ km}.$$

**3.4** From Equation (3.31a), we have

$$P(\bar{\nu}_e \to \bar{\nu}_x) = \sin^2(2\theta)\sin^2[\Delta(m^2c^4)L/(4\hbar c E)],$$

which for maximal mixing ($\theta = \pi/4$) gives

$$P(\bar{\nu}_e \to \bar{\nu}_x) = \sin^2[1.27\Delta(m^2c^4)L/E],$$

where $L$ is measured in metres, $E$ in MeV, $\Delta(m^2c^4)$ in (eV/c$^2$)$^2$ and

$$\Delta m^2 \equiv m^2(\bar{\nu}_e) - m^2(\bar{\nu}_x).$$

If $P(\bar{\nu}_e \to \bar{\nu}_e) = 0.9 \pm 0.1$, then at 95 % confidence level, $0.3 \geq P(\bar{\nu}_e \to \bar{\nu}_x) \geq 0$ and hence

$$0 \leq \Delta(m^2c^4) \leq 6.9 \times 10^{-3}\,(\text{eV/c}^2)^2.$$

**3.5** From the data given, the total number of nucleons is $N = 1.1 \times 10^{57}$ and hence $n = 7.7 \times 10^{38}$ km$^{-3}$. Also the mean energy of the neutrinos from reaction (3.38) is 0.26 MeV, so the cross-section is $\sigma = 1.8 \times 10^{-46}$ m$^2$. Thus $\lambda \approx 7 \times 10^6$ km, i.e. about 10 times the solar radius.

**3.6** (a) The quark compositions are: $D^- = dc$; $K^0 = d\bar{s}$; $\pi^- = d\bar{u}$ and since the dominant decay of a $c$-quark is $c \to s$, we have

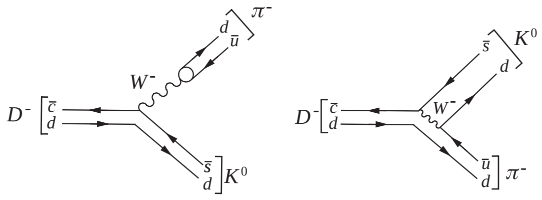


**Figure F.7**

(b) The quark compositions are: $\Lambda = sud$; $p = uud$ and since the dominant decay of an $s$-quark is $s \to u$, we have

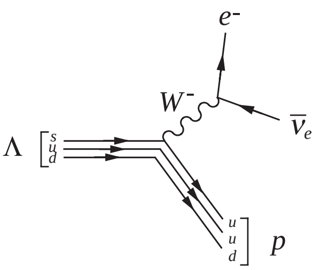


**Figure F.8**

**3.7** (a) This would be a baryon because $B = 1$ and the quark composition would be *ssb* which is allowed in the quark model.

(b) This would be a meson because $B = 0$, but would have to have both an $\bar{s}$ and a $\bar{b}$ quark. However $Q(\bar{s} + \bar{b}) = 2/3$, which is incompatible with the quark model and anyway combinations of two antiquarks are not allowed. Thus this combination is forbidden.

**3.8** 'Lowest-lying' implies that the internal orbital angular momentum between the quarks is zero. Hence the parity is $P = +$ and $\psi_{space}$ is symmetric. Since the Pauli principle requires the overall wavefunction to be antisymmetric under the interchange of any pair of like quarks, it follows that $\psi_{spin}$ is antisymmetric. Thus, any pair of like quarks must have antiparallel spins, i.e. be in a spin-0 state. Consider all possible baryon states *qqq*, where $q = u, d, s$. There are 6 combinations with a single like pair: $uud, uus, ddu, dds, ssu, ssd$, with the spin of *uu* etc equal to zero. Adding the spin of the third quark leads to 6 states with $J^P = \frac{1}{2}^+$. In principle there could be 6 combinations with all three quarks the same: $uuu, ddd, sss$, but in practice these do not occur because it is impossible to arrange all three spins in an antisymmetric way. Finally, there is 1 combination where all three quarks are different: *uds*. Here there are no restrictions from the Pauli principle, so the *ud* pair could have spin 0 or spin 1. Adding the spin of the $s$ quark leads to 2 states with $J^P = \frac{1}{2}^+$ and 1 with $J^P = \frac{3}{2}^+$.

Collecting the results, gives an octet of $J^P = \frac{1}{2}^+$ states and a singlet $J^P = \frac{3}{2}^+$ state. This is *not* what is observed in nature. In Chapter 5 we will see what additional assumptions have to be made to reproduce the observed spectrum.

**3.9** The ground state mesons all have $L = 0$ and $S = 0$. Therefore they all have $P = -1$. Only in the case of the neutral pion is their constituent quark and antiquark also particle and antiparticle. Thus $C$ is only defined for the $\pi^0$ and is $C = 1$. For the excited states, $L = 0$ still and thus $P = -1$ as for the ground states. However, the total spin of the constituent quarks is $S = 1$ and so for the $\rho^0$, the only state for which $C$ is defined, $C = -1$.

For the excited states, by definition there is a lower mass configuration with the same quark flavours. As the mass differences between the excited states and their ground states is greater than the mass of a pion, they can all decay by the strong interaction. In the case of the charged pions and kaons and the neutral kaon ground states, there are no lower mass configurations with the same flavour structure and so the only possibility is to decay via the weak interaction, with much longer lifetimes.

In the case of $\rho^0$ decay, the initial state has a total angular momentum of 1 and since the pions have zero spin, the $\pi\pi$ final state must have $L = 1$. While this is possible for $\pi^+\pi^-$, for the case of $\pi^0\pi^0$ it violates the Pauli Principle and so is forbidden.

**3.10** In the initial state, $S = -1$ and $B = 1$. To balance strangeness (conserved in strong interactions) in the final state $S(Y^-) = -2$ and to balance baryon number, $B(Y^-) = 1$. As charm and bottom for the initial state are both zero, these quantum numbers are zero for the $Y$. The quark content is therefore *dss*. In the decay, the strangeness of the $\Lambda$ is $S(\Lambda) = -1$ and so strangeness is not conserved. This is therefore a weak interaction and its lifetime will be the range $10^{-7}$ to $10^{-13}$ s.

**3.11** The quark composition is $\Sigma = uds$. Now

$$(\mathbf{S}_u + \mathbf{S}_d)^2 = \mathbf{S}_u^2 + \mathbf{S}_d^2 + 2\mathbf{S}_u \cdot \mathbf{S}_d = 2\hbar^2$$

and hence

$$\mathbf{S}_u \cdot \mathbf{S}_d = \hbar^2/4.$$

Then, from the general formula (3.91), setting $m_u = m_d = m$, we have

$$\begin{aligned} M_\Sigma &= 2m + m_s + \frac{b}{\hbar^2}\left[\frac{\mathbf{S}_u \cdot \mathbf{S}_d}{m^2} + \frac{\mathbf{S}_d \cdot \mathbf{S}_s + \mathbf{S}_u \cdot \mathbf{S}_s}{m m_s}\right] \\ &= 2m + m_s + \frac{b}{\hbar^2}\left[\frac{\mathbf{S}_u \cdot \mathbf{S}_d}{m^2} + \frac{\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 - \mathbf{S}_u \cdot \mathbf{S}_d}{m m_s}\right], \end{aligned}$$

which using

$$\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 = -3\hbar^2/4$$

from (3.96), gives

$$M_\Sigma = 2m + m_s + \frac{b}{4}\left[\frac{1}{m^2} - \frac{4}{m m_s}\right].$$

**3.12** The initial reaction is strong because it conserves all individual quark numbers. The $\Omega^-$ decay is weak because strangeness changes by one unit and the same is true for the decays of the $\Xi^0$, $K^+$ and $K^0$. The decay of the $\pi^+$ is also weak because it involves neutrinos and finally the decay of the $\pi^0$ is electromagnetic because only photons are involved.

**3.13** The Feynman diagram is shown in Figure F.9.

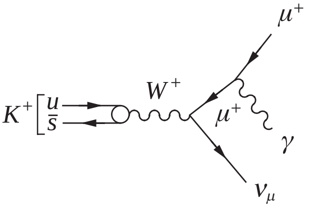


**Figure F.9**

The two vertices where the $W$ boson couples are weak interactions and have strengths $\sqrt{\alpha_W}$. The remaining vertex is electromagnetic and has strength $\sqrt{\alpha_{EM}}$. So the overall strength of the diagram is $\alpha_W\sqrt{\alpha_{EM}}$.

**3.14** Reactions (a), (d) and (f) conserve all quark numbers individually and hence are strong interactions. Reaction (e) violates strangeness and is a weak interaction. Reaction (c) conserves strangeness and involves photons and hence is an electromagnetic interaction. Reaction (b) violates both baryon number and electron lepton number and is therefore forbidden.

**3.15** The doublet of $S = +1$ mesons ($K^+$, $K^0$) has isospin $I = \frac{1}{2}$, with $I_3(K^+) = \frac{1}{2}$ and $I_3(K^0) = -\frac{1}{2}$. The triplet of $S = -1$ baryons ($\Sigma^+$, $\Sigma^0$, $\Sigma^-$) has $I = 1$, with $I_3 = 1, 0, -1$ for $\Sigma^+$, $\Sigma^0$ and $\Sigma^-$, respectively. Thus ($K^+$, $K^0$) is analogous to the $(p, n)$ isospin doublet and ($\Sigma^+$, $\Sigma^0$, $\Sigma^-$) is analogous to the $(\pi^+, \pi^0, \pi^-)$ isospin triplet. Hence, by analogy with Equations (3.61a,b),

$$\mathcal{M}(\pi^- p \to \Sigma^- K^+) = \tfrac{1}{3}\mathcal{M}_{3/2} + \tfrac{2}{3}\mathcal{M}_{1/2}, \mathcal{M}(\pi^- p \to \Sigma^0 K^0) = \tfrac{\sqrt{2}}{3}\mathcal{M}_{3/2} - \tfrac{\sqrt{2}}{3}\mathcal{M}_{1/2}$$

and

$$\mathcal{M}(\pi^+ p \to \Sigma^+ K^+) = \mathcal{M}_{3/2},$$

where $\mathcal{M}_{1/2,3/2}$ are the amplitudes for scattering in a pure isospin state $I = \frac{1}{2}, \frac{3}{2}$, respectively. Thus,

$$\begin{aligned}\sigma(\pi^+ p \to \Sigma^+ K^+) &: \sigma(\pi^- p \to \Sigma^- K^+) : \sigma(\pi^- p \to \Sigma^0 K^o) \\ &= \left|\mathcal{M}_{3/2}\right|^2 : \tfrac{1}{9}\left|\mathcal{M}_{3/2} + 2\mathcal{M}_{1/2}\right|^2 : \tfrac{2}{9}\left|\mathcal{M}_{3/2} - \mathcal{M}_{1/2}\right|^2\end{aligned}$$

**3.16** Under charge symmetry, $n(udd) \rightleftharpoons p(duu)$ and $\pi^+(u\bar{d}) \rightleftharpoons \pi^-(d\bar{u})$, and since the strong interaction is approximately charge symmetry, we would expect $\sigma(\pi^+ n) \approx \sigma(\pi^- p)$ at the same energy, with small violations due to electromagnetic effects and quark mass differences. However, $K^+(u\bar{s})$ and $K^-(s\bar{u})$ are not charge symmetric and so there is no reason why $\sigma(K^+ n)$ and $\sigma(K^- p)$ should be equal.

## Problems 4

**4.1** In an obvious notation,

$$\begin{aligned} E_{CM}^2 &= (E_e + E_p)^2 - (\mathbf{p}_e c + \mathbf{p}_p c)^2 \\ &= (E_e^2 - \mathbf{p}_e^2 c^2) + (E_p^2 - \mathbf{p}_p^2 c^2) + 2E_e E_p - 2\mathbf{p}_e \cdot \mathbf{p}_p c^2 \\ &= m_e^2 c^4 + m_p^2 c^4 + 2E_e E_p - 2\mathbf{p}_e \cdot \mathbf{p}_p c^2 \end{aligned}$$

At the energies of the beams, masses may be neglected and so with $p = |\mathbf{p}|$,

$$E_{CM}^2 = 2E_e E_p - 2p_e p_p c^2 \cos(\pi - \theta) = 2E_e E_p[1 - \cos(\pi - \theta)],$$

where $\theta$ is the crossing angle. Using the values given, gives $E_{CM} = 154$ GeV. In a fixed-target experiment, and again neglecting masses,

$$E_{CM}^2 = 2E_e E_p - 2\mathbf{p_e} \cdot \mathbf{p_p} c^2,$$

where

$$E_e = E_L, \quad E_p = m_p c^2, \quad \mathbf{p}_p = \mathbf{0}.$$

Thus, $E_{CM} = \left(2m_p c^2 E_L\right)^{1/2}$ and for $E_{CM} = 154$ GeV, this gives $E_L = 1.26 \times 10^4$ GeV.

**4.2** For constant acceleration, the ions must travel the length of the drift tube in half a cycle of the r.f. field. Thus, $L = \upsilon/2f$, where $\upsilon$ is the velocity of the ion. Since the energy is far less than the rest mass of the ion, we can use nonrelativistic kinematics to find $\upsilon$, i.e. $\upsilon = 4.01 \times 10^7$ m s$^{-1}$ and finally $L = 1$ m.

**4.3** A particle with mass $m$, charge $q$ and speed $\upsilon$ moving in a plane perpendicular to a constant magnetic field of magnitude $B$ will traverse a circular path with radius of curvature $r = m\upsilon/qB$ and hence the cyclotron frequency is $f = \upsilon/2\pi r = qB/2\pi m$. At each traversal the particle will receive energy from the r.f. field, so if $f$ is kept fixed, $r$ will increase (i.e. the trajectory will be a spiral). Thus if the final energy is $E$, the extraction radius will be $R = \sqrt{2mE}/qB$. To evaluate these expressions we use $q = 2e = 3.2 \times 10^{-19}$ C, together with

$$B = 0.8\ \mathrm{T} = 0.45 \times 10^{30}\ (\mathrm{MeV/c^2})\mathrm{s}^{-1}\ \mathrm{C}^{-1}$$

and thus $f = 6.15$ MHz and $R = 62.3$ cm.

**4.4** A particle with unit charge $e$ and momentum $p$ in the uniform magnetic field $B$ of the bending magnet will traverse a circular trajectory of radius $R$, given by $p = BR$. If $B$ is in Teslas, $R$ in metres and $p$ in GeV/c, then $p = 0.3\,BR$. Referring to Figure F.10, we have

$$\theta \approx L/R = 0.3\,LB/p \quad \text{and} \quad \Delta\theta = s/d = 0.3\,BL\Delta p/p^2.$$

Solving for $d$ using the data given, gives $d = 9.3$ m.

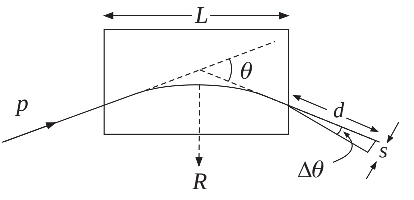


**Figure F.10**

**4.5** The Čerenkov condition is $\beta n \geq 1$. For the pion to give a signal, but not the kaon, we have $\beta_\pi n \geq 1 \geq \beta_K n$. The momentum is given by $p = m\upsilon\gamma$, so eliminating $\gamma$ gives

$$\beta = \upsilon/c = (1 + m^2c^2/p^2)^{-1/2}.$$

For $p = 20$ GeV/c, $\beta_\pi = 0.99997$, $\beta_K = 0.99970$, so the condition on the refractive index is

$$3 \times 10^{-4} \geq (n-1)/n \geq 3 \times 10^{-5}.$$

Using the largest value of $n = 1.0003$, we have

$$N = 2\pi\alpha\left(1 - \frac{1}{\beta_\pi^2 n^2}\right)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

as the number of photons radiated per metre, where $\lambda_1 = 400$ nm and $\lambda_2 = 700$ nm. Numerically, $N = 26.5$ photons/m and hence to obtain 200 photons requires a detector of length 7.5 m. (You could also use

$$N = 2\pi\alpha\left(1 - \frac{1}{\beta_\pi^2 n^2}\right)\left(\frac{\lambda_2 - \lambda_1}{\lambda^2}\right)$$

where $\lambda$ is the mean of $\lambda_1$ and $\lambda_2$, which would give 24.5 photons/m and a length of 8.2 m.)

**4.6** Luminosity may be calculated from Equation (4.7) for colliders, $L = n\, N_1\, N_2\, f/A$, where $n$ is the number of bunches, $N_{1,2}$ are the numbers of particles in each bunch, $A$ is the cross-sectional area of the beam and $f$ is its frequency. We have,

$$n = 12,\ N_1 = N_2 = 3 \times 10^{11},\ A = (2 \times 10^{-4})\ \text{cm}^2,$$
$$f = (3 \times 10^{10}/8\pi \times 10^5)\text{s}^{-1},$$

so finally $L = 6.44 \times 10^{31}\ \text{cm}^{-2}\text{s}^{-1}$.

**4.7** (a) The $b$ quarks are not seen directly, but instead, they ‘fragment’ (hadronize) to produce $B$-hadrons i.e. hadrons containing $b$ quarks. So one characteristic is the presence of hadrons with nonzero bottom quantum numbers. As these hadrons are unstable and the dominant decay of $b$ quarks is to $c$ quarks, a second characteristic is the presence of hadrons with nonzero values of the charm quantum number.

We need to observe the point where the $e^+e^-$ collision occurred and the point of origin of the decay products of the $B$ hadrons. The difference between these two is due to the lifetime of the $B$ hadrons. As the difference will be very small, precise position measurements are required. The daughter particles may be detected using a silicon vertex detector and a MWPC. In addition, any electrons from the decays could be detected by a MWPC or an electromagnetic calorimeter. The same is true for muons in the decay products, except they are not readily detected in the calorimeter as they are very penetrating. However, if a MWPC is placed behind a hadron calorimeter then one can be fairly confident that any particle detected is a muon, as everything else (except neutrinos) will have been stopped in the calorimeter.

(b) In the electronic decay mode, the electron can be measured in both a MWPC and an EM calorimeter. For high energies the better measurement is made in the calorimeter. The neutrino does not interact unless there is a very large mass of material (1000s of tons) and so its presence must be inferred by imposing conservation of energy and momentum. In a colliding beam machine, the original colliding particles have zero transverse momentum and a fixed energy. If one adds up all the energy and momentum of all the final state particles, then any imbalance compared to the initial system can be attributed to the neutrino.

For the muonic mode, the muon can be measured in the MWPC, but cannot be measured well in the calorimeter because it only ionizes to a very small extent. Since the muons only interact to a small extent they (along with neutrinos) are generally the only particles that emerge from a hadronic calorimeter. So if a signal is registered in a small MWPC placed behind a calorimeter then one can be confident that the particle is a muon.

**4.8** To be detected, the event must have $150^\circ < \theta < 30^\circ$, i.e. $|\cos\theta| < 0.866$. Setting $x = \cos\theta$, the fraction of events in this range is

$$f = \int_{-0.866}^{+0.866} \frac{\mathrm{d}\sigma}{\mathrm{d}x}\mathrm{d}x \Bigg/ \int_{-1.0}^{+1.0} \frac{\mathrm{d}\sigma}{\mathrm{d}x}\mathrm{d}x = 0.812.$$

The total cross-section is given by

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \mathrm{d}\cos\theta \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \frac{\alpha^2\hbar^2 c^2}{4E_{CM}^2} \int_{-1}^{+1} \left(1 + \cos^2\theta\right) \mathrm{d}\cos\theta.$$

Using $E_{CM} = 10$ GeV, gives

$$\sigma = 4\pi\alpha^2\hbar^2 c^2/3E_{CM}^2 = 0.866 \text{ nb}.$$

The rate of production of events is given by $L\sigma$ and since $L$ is a constant, the total number of events produced will be $L\sigma t = 86600$.

The $\tau^\pm$ decay too quickly to leave a visible track in the drift chamber. The $e^+$ and the $\mu^-$ will leave tracks in the drift chamber and the $e^+$ will produce a shower in the electromagnetic calorimeter. If it has enough energy, the $\mu^-$ will pass through the calorimeters and leave a signal in the muon chamber. There will no signal in the hadronic calorimeter.

**4.9** Referring to Figure F.11, the distance between two positions of the particle $\Delta t$ apart in time is $\upsilon\,\Delta t$. The wave fronts from these two positions have a difference in their distance travelled of $c\Delta t/n$.

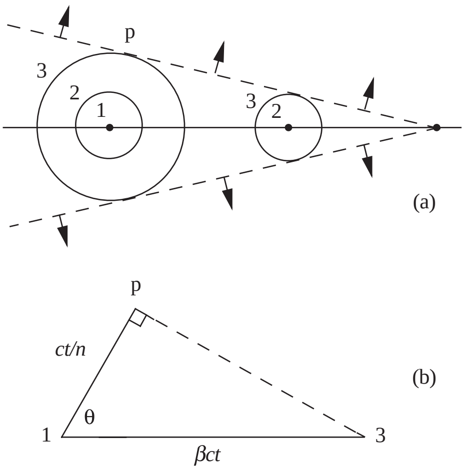


**Figure F.11**

These constructively interfere at an angle $\theta$, where

$$\cos\theta = \frac{c\,\Delta t/n}{\upsilon\,\Delta t} = \frac{1}{\beta\,n}.$$

The maximum value of $\theta$ corresponds to the minimum of $\cos\theta$ and hence the maximum of $\beta$. This occurs as $\beta \to 1$, when $\theta_{max} = \cos^{-1}(1/n)$. This value occurs in the ultra-relativistic or massless limit.
The quantity $\beta$ may be expressed as

$$\beta = pc/E = pc(p^2c^2 + m^2c^4)^{-1/2}.$$

Hence,

$$\cos\theta = \frac{1}{n}\frac{\sqrt{p^2c^2 + m^2c^4}}{pc},$$

which, after rearranging, gives

$$x \equiv (mc^2)^2 = p^2c^2(n^2\cos^2\theta - 1).$$

Differentiating this formula gives

$$\mathrm{d}x/\mathrm{d}\theta = -2p^2c^2n^2\cos\theta\sin\theta$$

and the error on $x$ is then given by $\sigma_x = |\mathrm{d}x/\mathrm{d}\theta|\,\sigma_\theta$. For very relativistic particles, the derivative can be approximated by using $\theta_{max}$, for which

$$\cos\theta_{max} = 1/n, \quad \sin\theta_{max} = \sqrt{n^2 - 1}/n.$$

Hence

$$\sigma_x \approx 2p^2c^2n^2 \frac{1}{n}\frac{\sqrt{n^2-1}}{n}\sigma_\theta = 2p^2c^2\sqrt{n^2-1}\,\sigma_\theta.$$

**4.10** The average distance between collisions of a neutrino and an iron nucleus is the mean free path $\lambda = 1/n\sigma_\nu$, where $n \approx \rho/m_p$ is the number of nucleons per cm$^3$. Using the data given, $n \approx 4.7 \times 10^{24}$ cm$^{-3}$ and $\sigma_\nu \approx 3 \times 10^{-36}$ cm$^2$, so that $\lambda \approx 7.1 \times 10^{10}$ cm. Thus if 1 in $10^9$ neutrinos is to interact, the thickness of iron required is 71 cm.

**4.11** Radiation energy losses are given by $-\mathrm{d}E/\mathrm{d}x = E/L_R$, where $L_R$ is the radiation length. This implies that $E = E_0 \exp(-x/L_R)$, where $E_0$ is the initial energy. Using the data given, gives $E = 1.51$ GeV. Radiation losses at fixed $E$ are proportional to $m^{-2}$, where $m$ is the mass of the projectile. Thus for muons, they are negligible at this energy.

**4.12** The total cross-section is $\sigma_{tot} = \sigma_{el} + \sigma_{cap} + \sigma_{fission} = 4 \times 10^2$ b and the attenuation is $\exp(-nx\sigma_{tot})$ where $nx = 10^{-1}N_A/A = 2.56 \times 10^{23}$ m$^{-2}$. Thus $\exp(-nx\sigma_{tot}) = 0.9898$, i.e. 1.02 % of the incident particles interact and of these the fraction that elastically scatter is given by the ratio of the cross-sections, i.e. $0.75 \times 10^{-4}$. Thus the intensity of elastically scattered neutrons is 0.765 s$^{-1}$ and finally the flux at 5 m is $2.44 \times 10^{-3}$ m$^{-2}$s$^{-1}$.

**4.13** The total centre-of-mass energy is given by $E_{\mathrm{CM}} \approx (2mc^2E_L)^{1/2} = 0.23$ GeV and so the cross-section is $\sigma = 1.64 \times 10^{-34}$ m$^2$. The interaction length is $l = 1/n\sigma$, where $n$ is the number density of electrons in the target. This is given by $n = \rho N_A Z\big/A$, where $N_A$ is Avogadro's number and for lead, $Z = 82$ and $A = 208$. Thus $n = 2.7 \times 10^{30}$ m$^{-3}$ and $l = 2.3 \times 10^3$ m.

**4.14** The target contains $n = 5.30 \times 10^{24}$ protons and so the total number of interactions per second is

$$N = n \times \text{flux} \times \sigma_{tot} = (5.30 \times 10^{24}) \times (2 \times 10^7) \times (40 \times 10^{-31}) = 424\ \mathrm{s}^{-1}.$$

There are thus 848 photons per sec produced from the target.

**4.15** For small $\upsilon$, the Bethe-Bloch formula may be written

$$S \equiv -\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{1}{\upsilon^2}\ln\left(\frac{2m_e\upsilon^2}{I}\right) \text{ with } \frac{\mathrm{d}S}{\mathrm{d}\upsilon} \propto \frac{2}{\upsilon^3}\left[1 - \ln\left(\frac{2m_e\upsilon^2}{I}\right)\right].$$

The latter has a maximum for $\upsilon^2 = eI/2m_e$. Thus for a proton in iron we can use $I = 10Z$ eV $= 260$ eV, so that

$$E_p = \tfrac{1}{2}m_p\upsilon^2 = m_p Ie/4m_e = 324\ \text{keV}.$$

**4.16** From (4.24), $E(r) = V\big/r\ln(r_c\big/r_a)$ and at the surface of the anode this is 4023 kV m$^{-1}$. Also, if $E_{threshold}(r) = 750$ kV m$^{-1}$, then from (4.24) $r = 0.107$ mm and so the distance to the anode is 0.087 mm. This contains 22 mean free paths and so assuming each collision produces an ion pair, the multiplication factor is $2^{22} = 4.2 \times 10^6 = 10^{6.6}$.

## Problems 5

**5.1** We have

$$m = \alpha + \beta + \gamma \geq n = \bar{\alpha} + \bar{\beta} + \bar{\gamma},$$

where the inequality is because baryon number $B > 0$. Using the values of the colour charges $I_3^C$ and $Y^C$ from Table 5.1, the colour charges for the state are:

$$I_3^C = (\alpha - \bar{\alpha})/2 - (\beta - \bar{\beta})/2$$

and

$$Y^C = (\alpha - \bar{\alpha})/3 + (\beta - \bar{\beta})/3 - 2(\gamma - \bar{\gamma})/3.$$

By colour confinement, both these colour charges must be zero for observable hadrons, which implies

$$\alpha - \bar{\alpha} = \beta - \bar{\beta} = \gamma - \bar{\gamma} \equiv p \quad \text{and hence} \quad m - n = 3p,$$

where $p$ is a non-negative integer. Thus the only combinations allowed by colour confinement are of the form

$$(3q)^p (q\bar{q})^n \quad (p, n \geq 0).$$

It follows that a state with the structure $qq$ is not allowed, as no suitable values of $p$ and $n$ can be found.

**5.2** The most general baryon colour wavefunction is

$$\chi_B^C = \alpha_1 r_1 g_2 b_3 + \alpha_2 g_1 r_2 b_3 + \alpha_3 b_1 r_2 g_3 + \alpha_4 b_1 g_2 r_3 + \alpha_5 g_1 b_2 r_3 + \alpha_6 r_1 b_2 g_3,$$

where the $\alpha_i$ $(i = 1, 2, \ldots, 6)$ are constants. If we apply the operator $\hat{F}_1$ to the first term and use the relations

$$\hat{F}_1 r = \tfrac{1}{2} g, \quad \hat{F}_1 g = \tfrac{1}{2} r, \quad \hat{F}_1 b = 0,$$

we have

$$\begin{aligned}\alpha_1 \hat{F}_1 (r_1 g_2 b_3) &= \alpha_1 (\hat{F}_1 r_1) g_2 b_3 + \alpha_1 r_1 (\hat{F}_1 g_2) b_3 + \alpha_1 r_1 g_2 (\hat{F}_1 b_3) \\ &= \frac{\alpha_1}{2} (g_1 g_2 b_3 + r_1 r_2 b_3)\,.\end{aligned}$$

Similar contributions are obtained by acting with $\hat{F}_1$ on the other terms in $\chi_B^C$ and collecting these together we obtain

$$\begin{aligned}\hat{F}_1\, \chi_B^C = {} & \frac{(\alpha_1 + \alpha_2)}{2} (g_1 g_2 b_3 + r_1 r_2 b_3) + \frac{(\alpha_3 + \alpha_4)}{2} (b_1 g_2 g_3 + b_1 r_2 r_3) \\ & + \frac{(\alpha_5 + \alpha_6)}{2} (g_1 b_2 g_3 + r_1 b_2 r_3)\,.\end{aligned}$$

This is only compatible with the confinement condition $\hat{F}_i \chi_B^C = 0$ for $i = 1$ if

$$\alpha_1 = -\alpha_2 \qquad \alpha_3 = -\alpha_4 \qquad \alpha_5 = -\alpha_6$$

The full set of such conditions leads to the antisymmetric form (5.2).

**5.3** (a)

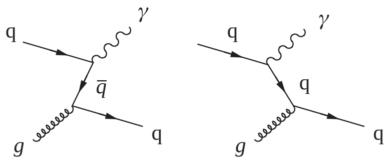


**Figure F.12**

(b)

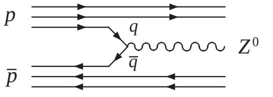


**Figure F.13**

(c)

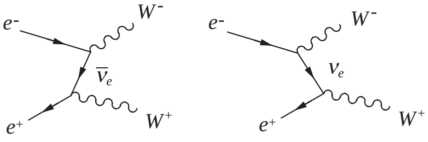


**Figure F.14**

(d)

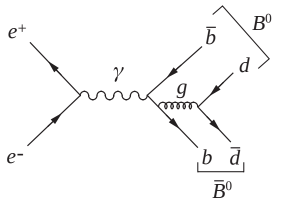


**Figure F.15**

There are several other alternative diagrams. For example, the photon could produce a $d\bar{d}$ pair and the gluon could be radiated by either the boson or the antiboson.

**5.4** The Feynman diagram is shown in Figure F.16.

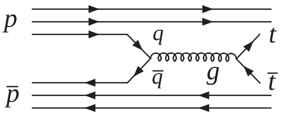


**Figure F.16**

The 4-momenta are:

$$P(p) = (E/c, \mathbf{p}) \quad \text{and} \quad P(\bar{p}) = (E/c, -\mathbf{p}),$$

with

$$P^2 = m^2c^2 = E^2/c^2 - \mathbf{p}^2 \quad \text{and} \quad m = m_p = m_{\bar{p}}.$$

Now $P(q) = (xE/c, x\mathbf{p})$ and $P(\bar{q}) = (xE/c, -x\mathbf{p})$ with $x = \frac{1}{6}$, so

$$E_{CM}^2 = x^2c^2\left[P(p) + P(\bar{p})\right]^2 = x^2\left(2m^2c^4 + 2E^2 + 2\mathbf{p}^2c^2\right).$$

Neglecting the masses of the proton and the antiproton at these energies, gives

$$E = 3E_{CM} \text{ and } p = 3 \times 350 = 1050 \text{ GeV/c}.$$

**5.5** Energy-momentum conservation gives,

$$W^2c^4 = [(E - E') + E_P]^2 - [(\mathbf{p} - \mathbf{p}) + \mathbf{P}]^2c^2 = \text{invariant mass squared of } X.$$

Using, $Q^2 = (\mathbf{p} - \mathbf{p}')^2 - (E - E')^2/c^2$ and $M^2c^4 = E_P^2 - \mathbf{P}^2c^2$, where $M$ is the mass of the proton, gives

$$W^2c^4 = -Q^2c^2 + M^2c^4 + 2E_P(E - E') - 2\mathbf{P} \cdot (\mathbf{p} - \mathbf{p}')c^2.$$

Also, $2M\nu \equiv W^2c^2 + Q^2 - M^2c^2$ and so, in the rest frame of the proton ($\mathbf{P} = \mathbf{0}$, $E_P = Mc^2$), $\nu = E - E'$. Since *some* energy must be transferred to the outgoing electron, it follows that $E \geq E'$, i.e. $\nu \geq 0$. Also, since the lightest state $X$ is the proton, $W^2 \geq M^2$. Thus,

$$2M\nu = Q^2 + (W^2 - M^2)c^2 \geq Q^2.$$

From the definition of $x$, it follows that $x \leq 1$. Finally, $x > 0$ because both $Q^2$ and $2M\nu$ are positive.

**5.6** By analogy with the QED formula, we have

$$\Gamma(3g) = 2(\pi^2 - 9)\alpha_s^6 m_c c^2/9\pi,$$

where $m_c \approx 1.5$ GeV/c$^2$ is the constituent mass of the $c$–quark. Evaluating this gives $\alpha_s = 0.31$. In the case of the radiative decay,

$$\Gamma(gg\gamma) = 2(\pi^2 - 9)\alpha_s^4\alpha^2 m_b c^2/9\pi,$$

where $m_b \approx 4.5$ GeV/c$^2$ is the constituent mass of the $b$–quark. Evaluating this gives $\alpha_s = 0.32$. (These values are a little too large because in practice $\alpha$ in the QED formulas should be replaced by $\frac{4}{3}\alpha_s$.)

**5.7** From (5.38a)

$$F_2^{\ell p}(x) = x\left[\tfrac{1}{9}\left(d+\bar{d}\right) + \tfrac{4}{9}\left(u+\bar{u}\right) + \tfrac{1}{9}\left(s+\bar{s}\right)\right]$$

and from (5.38b) and (5.39)

$$F_2^{\ell n}(x) = x\left[\tfrac{4}{9}\left(d+\bar{d}\right) + \tfrac{1}{9}\left(u+\bar{u}\right) + \tfrac{1}{9}\left(s+\bar{s}\right)\right],$$

so that

$$\int_0^1 \left[F_2^{ep}(x) - F_2^{en}(x)\right]\frac{\mathrm{d}x}{x} = \frac{1}{3}\int_0^1 \left[u(x)+\bar{u}(x)\right]\mathrm{d}x - \frac{1}{3}\int_0^1 \left[\mathrm{d}(x)+\bar{d}(x)\right]\mathrm{d}x.$$

But summing over all contributions we must recover the quantum numbers of the proton, i.e.

$$\int_0^1 [u(x)-\bar{u}(x)]\,\mathrm{d}x = 2; \quad \int_0^1 [d(x)-\bar{d}(x)]\,\mathrm{d}x = 1.$$

Eliminating the integrals over $u$ and $d$ gives the Gottfried sum rule.

**5.8** Substituting (5.22) into (5.23) and setting $N_C = 3$, gives

$$R = 3(1+\alpha_s/\pi)\sum e_q^2,$$

where $\alpha_s$ is given by (5.11) evaluated at $Q^2 = E_{CM}^2$ and the sum is over those quarks that can be produced in pairs at the energy considered. At 2.8 GeV the $u, d$ and $s$ quarks can contribute and at 15 GeV the $u, d, s, c$ and $b$ quarks can contribute. Evaluating $R$ then gives $R \approx 2.17$ at $E_{CM} = 2.8$ GeV and $R \approx 3.89$ at $E_{CM} = 15$ GeV. When $E_{CM}$ is above the threshold for $t\bar{t}$ production, $R$ rises to $R = 5(1+\alpha_s/\pi)$. (This ignores the effect of weak interactions, which are not negligible at these energies.)

**5.9** The Breit-Wigner form is given in Equation (1.80). Setting $S_e = \frac{1}{2}$ and $J = 1$ (the annihilation proceeds via photon exchange), gives

$$\sigma_f \equiv \sigma(e^+e^- \to f) = \frac{3\pi\hbar^2}{4q_e^2}\cdot\frac{\Gamma_{ee}\Gamma_f}{(E-Mc^2)^2+\Gamma^2/4},$$

where $M$ and $\Gamma$ are the mass and total width of $R$. Also $q_e \approx E_{CM}/2c = Mc^2/2c$, so

$$\sigma_f = \frac{3\pi(\hbar c)^2}{(Mc^2)^2}\cdot\frac{\Gamma_{ee}\Gamma_f}{(E-Mc^2)^2+\Gamma^2/4}.$$

Then, using the integral given,

$$\int \sigma_{\mu\mu}(E)\mathrm{d}E \approx \frac{6\pi^2(\hbar c)^2}{\Gamma(Mc^2)^2}\Gamma_{\mu\mu}^2 = 10\,\mathrm{nb}\cdot\mathrm{GeV}$$

and

$$\int \sigma_h(E)\mathrm{d}E \approx \frac{6\pi^2(\hbar c)^2}{\Gamma(Mc^2)^2}\Gamma_{\mu\mu}\Gamma_h = 300\,\mathrm{nb}\cdot\mathrm{GeV},$$

so that $\Gamma_h = 30\Gamma_{\mu\mu}$. Now, $\Gamma = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_h$, which using lepton universality becomes

$$\Gamma = \Gamma_h + 3\Gamma_{\mu\mu} = 33\Gamma_{\mu\mu}.$$

Finally, from the $\Gamma_h$ integrated cross-sections we have, using $\Gamma/\Gamma_h = 33/30$,

$$\Gamma_{\mu\mu} = 1.4\times 10^{-3}\ \mathrm{MeV}\ \text{and hence}\ \Gamma_h = 4.2\times 10^{-2}\ \mathrm{MeV}.$$

**5.10** For elastic scattering, $W^2 = M^2$ and so from the definition (5.24) $2M\nu = Q^2$ and hence

$$x \equiv Q^2/2M\nu = 1.$$

From the definition

$$Q^2 \equiv (\mathbf{p}-\mathbf{p}')^2 - (E-E')^2/c^2,$$

we have

$$Q^2 = \mathbf{p}^2 - 2\mathbf{p}\cdot\mathbf{p}' + \mathbf{p}'^2 - E^2/c^2 + 2EE'/c^2 - E'^2/c^2.$$

If we ignore the lepton masses, then $p \equiv |\mathbf{p}| = E/c$ and $p' \equiv \left|\mathbf{p}'\right| = E'/c$, so that

$$Q^2 = \frac{2EE'}{c^2}(1-\cos\theta).$$

Also, in the rest frame of the proton, $\nu = E - E'$, so substituting this and the expression for $Q^2$ into the relation $2M\nu = Q^2$ gives the result.

**5.11** A proton has the valance quark content $p = uud$. Thus from isospin invariance the $u$ quarks in the proton carry twice as much momentum as the $d$ quarks, which implies $a = 2b$. In addition, we are told that

$$\int_0^1 xF_u(x)\mathrm{d}x + \int_0^1 xF_d(x)\mathrm{d}x = \frac{1}{2}$$

Using the form of the quark distributions with $a = 2b$ gives $a = \frac{4}{3}$ and $b = \frac{2}{3}$.

**5.12** The peak value of the cross-section is where $E = M_W c^2$, i.e.

$$\sigma_{max} = \frac{\pi(\hbar c)^2(2/M_W c^2)^2\Gamma_{u\bar{d}}}{3\Gamma} = \frac{4}{3}\frac{\pi(\hbar c)^2}{(M_W c^2)^2}\mathrm{br}(W^+ \to u\bar{d}) = 84\ \mathrm{nb}.$$

The required integral is

$$\sigma_{p\bar{p}}(s) = \int_0^1\int_0^1 \sigma_{u\bar{d}}(E)u(x_u)\mathrm{d}(x_d)\mathrm{d}x_u\mathrm{d}x_d$$

where we have used $C$-invariance to relate the distribution functions for protons and antiprotons. In the narrow width approximation and using the quark distributions from Question 5.11,

$$\sigma_{p\bar{p}}(s) = C \int_0^1 \int_0^1 \frac{(1-x_u)^3}{x_u} \frac{(1-x_d)^3}{x_d} \, \delta\left[1 - \frac{x_u s}{(M_W c^2)^2} x_d\right] \mathrm{d}x_u \mathrm{d}x_d$$

where

$$C \equiv \frac{8\pi}{9} \frac{\Gamma_W}{M_W c^2} \sigma_{max} \quad \text{and} \quad E^2 = x_u x_d s.$$

Thus,

$$\sigma_{p\bar{p}}(s) = C \int_k^1 \frac{(1-x_u)^3}{x_u} \left(1 - \frac{k}{x_u}\right)^3 \mathrm{d}x_u,$$

where $k \equiv (M_W c^2)^2/s$ and the lower limit is because $k < x_u < 1$. The integral yields

$$\sigma_{p\bar{p}}(s) = \frac{8\pi}{9} \frac{\Gamma_W}{M_W c^2} \sigma_{max} \left[-(1 + 9k + 9k^2 + k^3) \ln k - \tfrac{11}{3} - 9k + 9k^2 + \tfrac{11}{3} k^3\right].$$

Evaluating this for $\sqrt{s} = 1$ TeV gives $k = 0.00646$ and $\sigma_{p\bar{p}} = 10.1$ nb which is about a factor of two larger than experiment.

## Problems 6

**6.1** A charged current weak interaction is one mediated by the exchange of a charged $W^{\pm}$ boson. A possible example is $n \to p + e^- + \bar{\nu}_e$. A neutral current weak is one mediated by a neutral $Z^0$ boson. An example is $\nu_\mu + p \to \nu_\mu + p$. Charged current weak interactions do not conserve the strangeness quantum number, whereas neutral current weak interaction do. For $\nu_\mu + e^- \to \nu_\mu + e^-$, the only Feynman diagram that conserves both $L_e$ and $L_\mu$ is shown in Figure F.17.

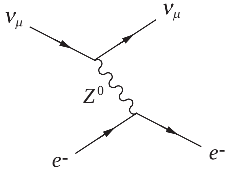


**Figure F.17**

which is a weak neutral current. However, for $\nu_e + e^- \to \nu_e + e^-$, there are two diagrams (see Figure F.18).

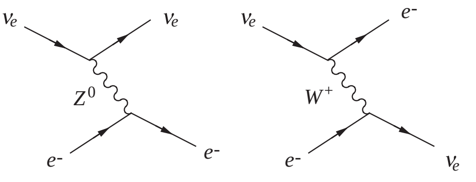


**Figure F.18**

Thus the reaction has both neutral and charged current components and is not unambiguous evidence for weak neutral currents.

**6.2** The lowest-order electromagnetic Feynman diagram is shown in Figure F.19.

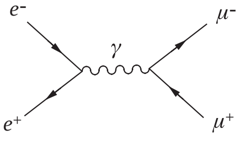


**Figure F.19**

The total cross-section is given by

$$\sigma = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\cos\theta \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi\alpha^2\hbar^2c^2}{4E_{CM}^2}\left[\cos\theta + \frac{1}{3}\cos^3\theta\right]_{-1}^{1}$$

$$= \frac{4\pi\alpha^2\hbar^2c^2}{3E_{CM}^2} = 0.44 \text{ nb}.$$

The lowest-order weak interaction diagram is shown in Figure F.20.

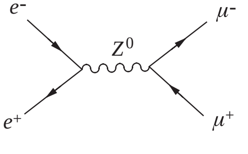


**Figure F.20**

With the addition of the weak interaction term,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{em} + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{wk} = \frac{\alpha^2\hbar^2c^2}{4E_{CM}^2}\left[1 + C_{wk}\cos\theta + \cos^2\theta\right].$$

Then, using

$$\sigma_F = C\int_0^1 \left[1 + C_{wk}\cos\theta + \cos^2\theta\right]\mathrm{d}\cos\theta$$

and

$$\sigma_B = C\int_{-1}^{0}\left[1 + C_{wk}\cos\theta + \cos^2\theta\right]\mathrm{d}\cos\theta,$$

where $C \equiv 2\pi\alpha^2\hbar^2c^2/4E_{CM}^2$, gives

$$\sigma_F = C\left[\frac{4}{3} + \frac{C_{wk}}{2}\right] \quad \text{and} \quad \sigma_B = C\left[\frac{4}{3} - \frac{C_{wk}}{2}\right]$$

and so

$$A_{FB} = \frac{C_{wk}}{2(4/3)}, \quad \text{i.e. } 8A_{FB} = 3C_{wk}.$$

**6.3** The Feynman diagram is shown in Figure F.21.

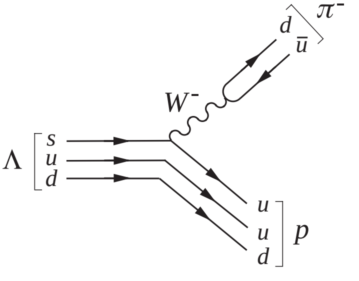


**Figure F.21**

The amplitude has two factors of the weak coupling $g_W$ and one $W$ propagator carrying a momentum $q$, i.e.

$$\text{amplitude} \propto \frac{g_W^2}{q^2c^2 - M_W^2c^4} \propto \frac{g_W^2}{M_W^2},$$

because $qc \approx M_\Lambda c^2 \ll M_W c^2$. Now,

$$\Gamma(\Lambda \to p\pi^-) \propto (\text{amplitude})^2 \propto g_W^4/M_W^4$$

and so doubling $g_W$ and reducing $M_W$ by a factor of four will increase the rate by a factor

$$\left[2^4\right]\Big/\left[(1/4)^4\right] = 4096.$$

**6.4** The most probable energy is given by

$$\frac{\mathrm{d}}{\mathrm{d}E_e}\left(\frac{\mathrm{d}\omega}{\mathrm{d}E_e}\right) = 0,$$

which gives

$$\frac{2G_F^2m_\mu^2}{(2\pi)^3(\hbar c)^6}\left(2E_e - \frac{4E_e^2}{m_\mu c^2}\right) = 0, \quad \text{i.e. } E_e = m_\mu c^2/2.$$

When $E_e \approx m_\mu c^2/2$, the electron has its maximum energy and the two neutrinos must be recoiling in the opposite direction. Only left-handed particles (and right-handed antiparticles) are produced in weak interactions. Since the masses of all particles are neglected, states of definite handiness are also states of definite helicity, so the orientations of the momenta and spins are therefore as shown:

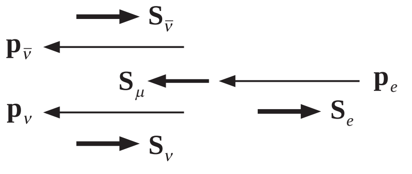


**Figure F.22**

Integrating the spectrum gives

$$\Gamma = \frac{2G_F^2(m_\mu c^2)^2}{(2\pi)^3(\hbar c)^6} \int_0^{m_\mu c^2/2} \left[E_e^2 - \frac{4E_e^3}{3m_\mu c^2}\right] \mathrm{d}E_e = \frac{G_F^2(m_\mu c^2)^5}{192\pi^3(\hbar c)^6}.$$

Numerically, $\Gamma \approx 3.0 \times 10^{-19}$ GeV, which gives a lifetime $\tau = \hbar/\Gamma \approx 2.2 \times 10^{-6}$ s.

**6.5** (a) In addition to the decay $b \to c + e^- + \bar{\nu}_e$, there are two other leptonic decays ($\ell = \mu^-, \tau^-$) and by lepton universality they will all have equal decay rates. There are also hadronic decays of the form $b \to c + X$ where $Q(X) = -1$. Examining the allowed $Wq\bar{q}$ vertices using lepton-quark symmetry shows that the only forms that $X$ can have, if we ignore Cabibbo-suppressed modes, are $d\bar{u}$ and $s\bar{c}$. Each of these hadronic decays has a probability three times that of a leptonic decay because the quarks exist in three colour states. Thus, there are effectively 6 hadronic channels and 3 leptonic ones. So finally,

$$BR(b \to c + e^- + \bar{\nu}_e) = 1/9.$$

(b) The argument is similar to that of (a) above. Thus, in addition to the decay $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$, there is also the leptonic decay $\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau$ with equal probability and the hadronic decays $\tau^- \to \nu_\tau + X$. In principle, $X = d\bar{u}$ and $s\bar{c}$, but the latter is not allowed because $m_s + m_c > m_\tau$. So the only allowed hadronic decay is $\tau^- \to d + \bar{u} + \nu_\tau$ with a relative probability of 3 because of colour. So finally,

$$BR(\tau^- \to e^- + \bar{\nu}_e + \nu_\tau) = 1/5.$$

(The measured rate is 0.18, but we have neglected kinematic corrections.)

**6.6** For neutrinos, $g_R(\nu) = 0$ and $g_L(\nu) = \frac{1}{2}$. So,

$$\Gamma_{\nu_e} = \Gamma_{\nu_\mu} = \Gamma_{\nu_\tau} = \Gamma_0/4,$$

where

$$\Gamma_0 = \frac{G_F M_Z^3 c^6}{3\pi\sqrt{2}(\hbar c)^3} = 668\ \text{MeV}.$$

Thus the partial width for decay to neutrino pairs is $\Gamma_\nu = 501$ MeV. For quarks,

$$g_R(u, c, t) = -\tfrac{1}{6} \text{ and } g_L(u, c, t) = \tfrac{1}{3}.$$

Thus, $\Gamma_u = \Gamma_c = \frac{10}{72}\Gamma_0$. Also,

$$g_R(d, s, b) = \tfrac{1}{12} \quad \text{and} \quad g_L(b, s, d) = -\tfrac{5}{12}.$$

Thus, $\Gamma_d = \Gamma_s = \Gamma_b = \frac{13}{72}\Gamma_0$. Finally,

$\Gamma_q = \sum_i \Gamma_i$, where $i = u, c, d, s, b$ – no top quark because $2M_t > M_Z$. So,

$$\Gamma_q = \left(\frac{3 \times 13}{72} + \frac{2 \times 10}{72}\right)\Gamma_0 = \frac{59}{72}\Gamma_0 = 547\ \text{MeV}.$$

Hadron production is assumed to be equivalent to the production of $q\bar{q}$ pairs followed by fragmentation with probability unity. Thus $\Gamma_{hadron} = 3\Gamma_q$, where the factor of three is because each quark exists in one of three colour states. Thus $\Gamma_{hadron} = 1641$ MeV.

If there are $N_\nu$ generations of neutrinos with $M_\nu < M_Z/2$, so that $Z^0 \to \nu\bar{\nu}$ is allowed, then

$$\Gamma_{tot} = \Gamma_{hadron} + \Gamma_{lepton} + N_\nu \Gamma_{\nu\bar{\nu}}$$

where $\Gamma_{\nu\bar{\nu}}$ is the width to a specific $\nu\bar{\nu}$ pair. Thus

$$\begin{aligned} N_\nu &= \frac{\Gamma_{tot} - \Gamma_{hadron} - \Gamma_{lepton}}{\Gamma_{\nu\bar{\nu}}} \\ &= \frac{(2490 \pm 7) - (1738 \pm 12) - (250 \pm 2)}{167} = 3.01 \pm 0.05, \end{aligned}$$

which rules out values of $N_\nu$ greater than 3.

**6.7** The quark compositions are: $D^0 = c\bar{u}$; $K^- = s\bar{u}$; $\pi^+ = u\bar{d}$. Since preferentially $c \to s$, we have

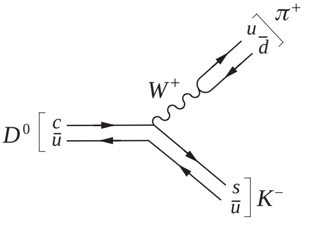


**Figure F.23**

i.e. a lowest-order charge current weak interaction. However, for $D^+ \to K^0 + \pi^+$, we have

$$D^+ = c\bar{d}; \quad K^0 = d\bar{s}; \quad \pi^+ = u\bar{d}.$$

Thus we could arrange $c \to d$ via $W$ emission and the $W^+$ could then decay to $u\bar{d}$, i.e. $\pi^+$. However, this would leave the $\bar{d}$ quark in the $D^+$ to decay to an $\bar{s}$ quark in the $K^0$ which is not possible as they both have the same charge.

**6.8** The relevant Feynman diagrams are shown in Figure F.24.

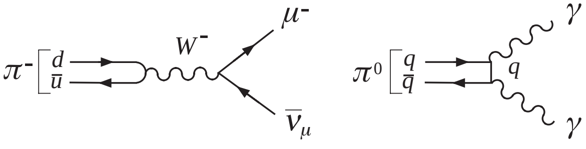


**Figure F.24**

In the case of the charged pion, there are two vertices of strength $\sqrt{\alpha_W}$, and there will be a propagator

$$\frac{1}{Q^2 + M_W^2 c^2} \approx \frac{1}{M_W^2 c^2},$$

because the momentum transfer (squared) $Q^2$ carried by the $W$ is very small compared to the mass of the $W$. Thus the decay rate will be proportional to

$$\left( \frac{\sqrt{\alpha_W}\sqrt{\alpha_W}}{M_W^2} \right)^2 = \frac{\alpha_W^2}{M_W^4}.$$

In the case of the neutral pion, there are two vertices of strength $\sqrt{\alpha_{em}}$, but no propagator with a $W$-mass factor. (The effective propagator will depend on the mass of the quark.) Thus the decay rate will be proportional to $\alpha_{em}^2$ and since $\alpha_{em} \approx \alpha_W$, the decay rate for the charged pion will be much smaller than that for the neutral decay, i.e. the lifetime of the $\pi^0$ will be much shorter.

**6.9** The two Feynman diagrams are shown in Figure F.25.

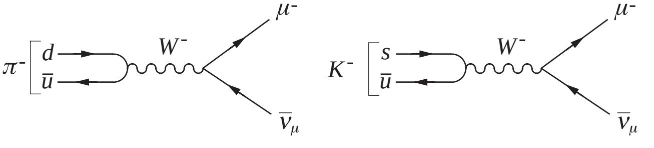


**Figure F.25**

Using lepton-quark symmetry and the Cabibbo hypothesis, the two hadron vertices are given by

$$g_{udW} = g_W \cos\theta_C \quad \text{and} \quad g_{usW} = g_W \sin\theta_C.$$

So, if we ignore kinematic differences and spin effects, and use $\theta_C = 13^0$, we would expect the ratio of decay rates is given by

$$R = \frac{\text{Rate }(K^- \to \mu^- + \bar{\nu}_\mu)}{\text{Rate }(\pi^- \to \mu^- + \bar{\nu}_\mu)} \propto \frac{g_{usW}^2}{g_{udW}^2} = \tan^2\theta_C \approx 0.05$$

The measured ratio is actually about 1.3, which shows the importance of the neglected effects. For example, the $Q$–value for the kaon decay is about 10 times that for pion decay.

**6.10** To a first approximation the difference in the two decay rates is due to two effects. Firstly, $\Sigma^- \to n + e^- + \bar{\nu}_e$ has $|\Delta S| = 1$ and hence is proportional to $\sin^2\theta_C$, where $\theta_C \approx 13^0$ is the Cabbibo angle, whereas $\Sigma^- \to \Lambda + e^- + \bar{\nu}_e$ has $|\Delta S| = 0$ and is proportional to $\cos^2\theta_C$. Secondly, the $Q$–values are different for the two reactions. Thus, using Sargent's Rule,

$$R \approx \frac{\sin^2\theta_C}{\cos^2\theta_C}\left(\frac{Q_{\Sigma n}}{Q_{\Sigma\Lambda}}\right)^5 \approx 0.053\left(\frac{258}{82}\right)^5 = 16.6.$$

(The experimental value is 17.8.)

**6.11** The required number of events produced must be 20,000, taking account of the detection efficiency. If the cross-section is 60 fb $= 6 \times 10^{-38}$ cm$^2$, then the integrated luminosity required is

$$2 \times 10^4 / 6 \times 10^{-38} = (1/3) \times 10^{42}\ \text{cm}^{-2}$$

and hence the instantaneous luminosity must be $3.3 \times 10^{34}$ cm$^{-2}$s$^{-1}$.

The branching ratio for $Z^0 \to b\bar{b}$ is found from the partial widths to be 15 %. Thus, if $b$ quarks are detected, the much greater branching ratio for $H \to b\bar{b}$ will help distinguish this decay from the background of $Z^0 \to b\bar{b}$.

**6.12** By 'adding' an $I = \frac{1}{2}$ particle to the initial state we can assume isospin invariance holds. Consider $\Xi^- + S^0 \to \Lambda + \pi^-$. The final state is $|I = 1, I_3 = -1\rangle$ and so is the initial state because $I_3(S^0) = -\frac{1}{2}$. Thus the transition is pure $I = 1$ and the rate is $|M_1|^2$. For $\Xi^0 + S^0 \to \Lambda + \pi^0$, the final state is again pure $I = 1$ but with $I_3 = 0$. However the initial state is an equal mixture of $I = 0$ and $I = 1$, i.e.

$$\left|\Xi^- S^0\right\rangle = \tfrac{1}{\sqrt{2}}\,|I = 1, I_3 = 0\rangle \pm \tfrac{1}{\sqrt{2}}\,|I = 0, I_3 = 0\rangle$$

and so the rate is $\frac{1}{2}|M_1|^2$. Thus $R = 2$. (The measured value is about 1.8.)

**6.13** Integrating the differential cross-sections over $y$ (from 0 to 1) gives for a spin-$\frac{1}{2}$ target with a specific quark distribution

$$\frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = \left[\int_0^1 \left[g_L^2 + g_R^2(1-y)^2\right]\mathrm{d}y\right]\left[\int_0^1 \mathrm{d}y\right]^{-1} = g_L^2 + \tfrac{1}{3}g_R^2$$

and

$$\frac{\sigma^{NC}(\overline{\nu})}{\sigma^{CC}(\overline{\nu})} = \left[\int_0^1 \left[g_L^2(1-y)^2 + g_R^2\right] \mathrm{d}y\right] \left[\int_0^1 (1-y)^2 \mathrm{d}y\right]^{-1} = g_L^2 + 3g_R^2.$$

For an isoscalar target, we must add the contributions for $u$ and $d$ quarks in equal amounts, i.e.

$$\frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)}(\text{isoscalar}) = g_L^2(u) + \tfrac{1}{3}g_R^2(u) + g_L^2(d) + \tfrac{1}{3}g_R^2(d)$$

and

$$\frac{\sigma^{NC}(\overline{\nu})}{\sigma^{CC}(\overline{\nu})}(\text{isoscalar}) = g_L^2(u) + 3g_R^2(u) + g_L^2(d) + 3g_R^2(d)$$

Substituting for the couplings finally gives for an isoscalar target

$$\frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = \tfrac{1}{2} - \sin^2\theta_W + \tfrac{20}{27}\sin^4\theta_W, \quad \frac{\sigma^{NC}(\overline{\nu})}{\sigma^{CC}(\overline{\nu})} = \tfrac{1}{2} - \sin^2\theta_W + \tfrac{20}{9}\sin^4\theta_W$$

**6.14** The decay in question is $B \to D\ell\nu_\ell$ and the appropriate matrix element is $V_{cb}$. The analogous formula for the lifetime $\tau_B$ is therefore

$$\frac{1}{\tau_B} = \frac{G_F^2(m_b c^2)^5}{192\pi^3\hbar(\hbar c)^6}\frac{|V_{cb}|^2}{B(B \to D\ell\nu)},$$

where $m_b$ is the mass if the bottom quark, and $B(B \to D\ell\nu)$ is the branching ratio.

Using $B = 0.11$, $\tau_B = 1.6 \times 10^{-12}$ s and $m_b \approx 4.5$ GeV/c$^2$ from Appendix E, and $G_F/(\hbar c)^3 = 1.17 \times 10^{-5}$ GeV$^{-2}$, gives $|V_{cb}| = 0.032$. (The actual value is 0.04.)

**6.15** Direct substitution, with $A_0$ real, gives

$$\mathcal{M}(K_S^0 \to \pi^0\pi^0) \approx \frac{2N}{\sqrt{3}}\left[\sqrt{2}e^{i\delta_2}\ \mathrm{Re}A_2 - e^{i\delta_0}A_0\right]$$

and

$$\mathcal{M}(K_L^0 \to \pi^0\pi^0) \approx \frac{2N}{\sqrt{3}}\left[i\sqrt{2}e^{i\delta_2}\ \mathrm{Im}A_2 - \varepsilon e^{i\delta_0}A_0\right]$$

where second-order terms have been neglected. Thus

$$\eta_{00} = \frac{\varepsilon - i\sqrt{2}e^{i\Delta}\ \mathrm{Im}A_2/A_0}{1 - \sqrt{2}e^{i\Delta}\ \mathrm{Re}A_2/A_0} \approx \varepsilon - i\sqrt{2}\exp(i\Delta)\frac{\mathrm{Im}A_2}{A_0},$$

where $\Delta \equiv \delta_2 - \delta_0$ and again second-order terms have been neglected.

## Problems 7

**7.1** For the ${}^7_3$Li nucleus, $Z = 3$ and $N = 4$. Hence, from Figure 7.4, the configuration is:

$$\text{protons:} \quad \left(1s_{1/2}\right)^2 \left(1p_{3/2}\right)^1; \quad \text{neutrons:} \quad \left(1s_{1/2}\right)^2 \left(1p_{3/2}\right)^2$$

By the pairing hypothesis, the two neutrons in the $1p_{3/2}$ sub-shell will have a total orbital angular momentum and spin $\mathbf{L} = \mathbf{S} = \mathbf{0}$ and hence $\mathbf{J} = \mathbf{0}$. Therefore they will not contribute to the overall nuclear spin, parity or magnetic moment. These will be determined by the quantum numbers of the unpaired proton in the $1p_{3/2}$ subshell. This has $J = \frac{3}{2}$ and $l = 1$, hence for the spin-parity we have $J^P = \frac{3}{2}^-$. The magnetic moment is given by

$$\mu = jg_{\text{proton}} = j + 2.3 \text{ (since } j = l + \tfrac{1}{2}) = 1.5 + 2.3 = 3.8 \text{ nuclear magnetons.}$$

If only protons are excited, the two most likely excited states are:

$$\text{protons:} \quad \left(1s_{1/2}\right)^2 \left(1p_{1/2}\right)^1; \quad \text{neutrons:} \quad \left(1s_{1/2}\right)^2 \left(1p_{3/2}\right)^2,$$

which corresponds to exciting a proton from the $p_{3/2}$ subshell to the $p_{1/2}$ subshell, and

$$\text{protons:} \quad \left(1s_{1/2}\right)^{-1} \left(1p_{3/2}\right)^2; \quad \text{neutrons:} \quad \left(1s_{1/2}\right)^2 \left(1p_{3/2}\right)^2,$$

which corresponds to exciting a proton from the $s_{1/2}$ subshell to the $p_{3/2}$ subshell.

**7.2** A state with quantum number $j = l \pm \frac{1}{2}$ can contain a maximum number $N_j = 2(2j + 1)$ nucleons. Therefore, if $N_j = 16$ if follows that $j = \frac{7}{2}$ and $l = 3$ or 4. But we know that the parity is odd and since $P = (-1)^l$, it follows that $l = 3$.

**7.3** The configuration of the ground state is:

$$\text{protons:} \quad (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2}); \quad \text{neutrons:} \quad (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$$

To get $j^P = \frac{1}{2}^-$, one could promote a $p_{1/2}$ proton to the $d_{5/2}$ shell, giving:

$$\text{protons:} \quad (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^{-1}(1d_{5/2})^2.$$

Then by the pairing hypothesis, the two $d_{5/2}$ protons could combine to give $j^P = 0^+$, so that the total spin-parity would be determined by the unpaired $p_{1/2}$ proton, i.e. $j^P = \frac{1}{2}^-$. Alternatively, one of the $p_{3/2}$ neutrons could be promoted to the $d_{5/2}$ shell, giving

$$\text{neutrons:} \quad (1s_{1/2})^2(1p_{3/2})^{-1}(1p_{1/2})^2(1d_{5/2})$$

and the $d_{5/2}$ proton and $d_{5/2}$ neutron and could combine to give $j^P = 2^+$, so that when this combines with the single unpaired $j^P = \frac{3}{2}^-$ neutron (the other two $j^P = \frac{3}{2}^-$ neutrons would pair to given $j^P = 0^+$), the overall spin-parity is $j^P = \frac{1}{2}^-$. There are other possibilities.

**7.4** For ${}^{93}_{41}$Nb, $Z = 41$ and $N = 52$. From the filling diagram Figure 7.4, the configuration is predicted to be:

$$\text{proton:} \quad \ldots(2p_{3/2})^4(1f_{5/2})^6(2p_{1/2})^2(1g_{9/2})^1; \quad \text{neutron:} \quad \ldots(2d_{5/2})^2.$$

So $l = 4$, $j = \frac{9}{2} \Rightarrow j^P = \frac{9}{2}^+$ (which agrees with experiment). The magnetic dipole moment follows from the expression for $j_{\text{proton}}$ in (7.31) with $j = l + \frac{1}{2}$, i.e. $\mu = (j + 2.3)\mu_N = 6.8\mu_N$. (The measured value is $6.17\mu_N$.)

For ${}^{33}_{16}\text{S}$, $Z = 17$ and $N = 17$. From the filling diagram Figure 7.4, the configuration is predicted to be:

$$\text{proton:} \quad \cdots(1d_{5/2})^6(2s_{1/2})^2; \qquad \text{neutron:} \quad \cdots(1d_{5/2})^7(2s_{1/2})^2(1d_{3/2})^1.$$

So $l = 2$, $j = \frac{3}{2} \Rightarrow j^P = \frac{3}{2}^+$ (which agrees with experiment). The magnetic dipole moment follows from the expression for $j_{\text{neutron}}$ in (7.31) with $j = l - \frac{1}{2}$, i.e. $\mu = [(1.9j)/(j+1)]\mu_N = 1.14\mu_N$. (The measured value is $0.64\mu_N$.)

**7.5** From (7.32),

$$eQ = \int \rho(2z^2 - x^2 - y^2)\mathrm{d}\tau$$

with $\rho = Ze/(\frac{4}{3}\pi b^2 a)$ and the integral is through the volume of the spheroid $(x^2 + y^2)/b^2 + z^2/a^2 \leq 1$. The integral can be transformed to one over the volume of a sphere by the transformations $x = bx'$, $y = by'$ and $z = az'$. Then

$$Q = \frac{3Z}{4\pi} \iiint \mathrm{d}x'\ \mathrm{d}y'\ \mathrm{d}z'\ (2a^2 z'^2 - b^2 x'^2 - b^2 y'^2).$$

But

$$\iiint x'^2 \mathrm{d}x'\ \mathrm{d}y'\ \mathrm{d}z' = \frac{1}{3}\int_0^1 r'^2 4\pi r'^2 \mathrm{d}r' = \frac{4\pi}{15},$$

and similarly for the other integrals. Thus, by direct substitution, $Q = \frac{2}{5}Z(a^2 - b^2)$.

**7.6** From Question 7.5 we have $Q = \frac{2}{5}Z(a^2 - b^2)$ and using $Z = 67$ this gives $a^2 - b^2 = 13.1\ \text{fm}^2$. Also, from Equation (2.32) we have $A = \frac{4}{3}\pi ab^2\rho$, where $\rho = 0.17\ \text{fm}^{-3}$ is the nuclear density. Thus, $ab^2 = 231.7\ \text{fm}^3$. The solution of these two equations gives $a \approx 6.85$ fm and $b \approx 5.82$ fm.

**7.7** From Equation (7.53),

$$t_{1/2} = \ln 2/\lambda = aR \ln 2 \exp(G),$$

where $a$ is a constant formed from the frequency and the probability of forming alpha particles in the nucleus. Thus

$$t_{1/2}(\text{Th}) = t_{1/2}(\text{Cf})R(\text{Th})\exp[G(\text{Th}) - G(\text{Cf})]R(\text{Cf}).$$

The Gamow factors may be calculated from the data given. Some intermediate quantities are:

$$R = 9.268\,\text{fm}\,(\text{Th}), \quad 9.439\,\text{fm}\,(\text{Cf}),$$

(using $R = 1.21(A^{1/3} + 4^{1/3})$ and recalling that $(Z, A)$ refers to the daughter nucleus). These give

$$G = 66.4\ (\text{Th}), \quad 54.8\ (\text{Cf})$$

and

$$t_{1/2}(\mathrm{Th}) = 0.98e^{11.6} \times t_{1/2}(\mathrm{Cf}) \approx 3.9\,\mathrm{yr}.$$

(The measured value is 1.9 yr.)

**7.8** The $J^P$ values of the $\Sigma^0$ and the $\Lambda$ are both $\frac{1}{2}^+$ (see Chapter 3), so the photon has $L = 1$ and as there is no change of parity the decay proceeds via an M1 transition. The $\Delta^0$ has $J^P = \frac{3}{2}^+$ and again there is no parity change. Therefore both M1 and E2 multipoles could be involved, with M1 dominant (see Section 7.8.2). If we assume that the reduced transition probabilities are equal in the two cases, then from Equation (7.89), in an obvious notation,

$$\tau(\Sigma^0) = \left[\frac{E_\gamma(\Delta^0)}{E_\gamma(\Sigma^0)}\right]^3 \tau(\Delta^0).$$

Thus,

$$\tau(\Sigma^0) = (292/77)^3 \times (0.6 \times 10^{-23})/0.0056 = 5.8 \times 10^{-20}\ \mathrm{s}.$$

(The measured value is $(7.4 \pm 0.7) \times 10^{-20}$ s.)

**7.9** In the centre-of-mass system, the threshold for ${}^{34}\mathrm{S} + p \to n + {}^{34}\mathrm{Cl}$ is $6.45 \times (34/35) = 6.27$ MeV. Correcting for the neutron-proton mass difference gives the Cl-S mass difference as 5.49 MeV and since in the positron decay ${}^{34}\mathrm{Cl} \to {}^{34}\mathrm{S} + e^+ + \nu_e$, the energy released is

$$Q = M(A, Z) - M(A, Z-1) - 2m_e,$$

the maximum positron energy is 4.47 MeV.

**7.10** We need to calculate the fraction

$$F \equiv \left[\int_{T_0-\Delta}^{T_0} I(T)\,\mathrm{d}T\right]\left[\int_0^{T_0} I(T)\,\mathrm{d}T\right]^{-1}$$

where $\Delta = 5$ eV and $I(T) = T^{1/2}(T_0 - T)^2$ with $T_0 = 18.6$ keV. Evaluating the denominator gives $16T_0^{7/2}/105$ and using the integral given, the numerator is $T_0^{1/2}\Delta^3/3$. Thus

$$F = \frac{T_0^{1/2}\Delta^3}{3} \cdot \frac{105}{16T_0^{7/2}} = 2.19\left(\frac{\Delta}{T_0}\right)^3 = 4.25 \times 10^{-11}.$$

**7.11** The mean energy $\bar{T}$ is defined by

$$\bar{T} \equiv \left[\int_0^{T_0} T\ \mathrm{d}\omega(T)\right]\left[\int_0^{T_0} \mathrm{d}\omega(T)\right]^{-1}.$$

The integrals are:

$$\int T^{3/2}(T_0 - T)^2 \mathrm{d}T = \tfrac{2}{315} T^{5/2}\left[63T_0^2 - 90T_0 T + 35T^2\right]$$

and

$$\int T^{1/2}(T_0 - T)^2 \mathrm{d}T = \tfrac{2}{105} T^{3/2}\left[35T_0^2 - 42T_0T + 15T^2\right].$$

Substituting the limits gives $\overline{T} = \frac{1}{3}T_0$, as required.

**7.12** The possible transitions are as follows:

| Initial | Final | $L$ | $\Delta P$ | Multipoles |
|---|---|---|---|---|
| $\frac{3}{2}^-$ | $\frac{5}{2}^-$ | 1, 2, 3, 4 | No | M1, E2, M3, ... |
| $\frac{3}{2}^-$ | $\frac{1}{2}^-$ | 1, 2 | No | M1, E2 |
| $\frac{5}{2}^-$ | $\frac{1}{2}^-$ | 2, 3 | No | E2, M3 |

From Figure 7.13, the dominant multipole for a fixed transition energy will be M1 for the $\frac{3}{2}^- \to \frac{5}{2}^-$ and $\frac{3}{2}^- \to \frac{1}{2}^-$ transitions and E2 for the $\frac{5}{2}^- \to \frac{1}{2}^-$ transition. Thus we need to calculate the rate for a M1 transition with $E_\gamma = 178$ keV. This can be done using Equation (7.89) and gives $\tau_{1/2} \approx 3.9 \times 10^{-12}$ s. The measured value is $3.5 \times 10^{-10}$ s, which confirms that the Weisskopf approximation is not very accurate.

**7.13** Set $L = 3$ in (7.88a), substitute the result into (7.86) and use $\Gamma_\gamma = \hbar T$ to give $\Gamma_\gamma(\mathrm{E3}) = (2.3 \times 10^{-14})E_\gamma^7 A^2$ eV, where $E_\gamma$ is expressed in MeV.

## Problems 8

**8.1** To balance the number of protons and neutrons, the fission reaction must be

$$n + {}^{235}_{92}\mathrm{U} \to {}^{92}_{37}\mathrm{Rb} + {}^{140}_{55}\mathrm{Cs} + 4n.$$

The energy released is the differences in binding energies of the various nuclei, because the mass terms in the SEMF cancel out. We have, in an obvious notation,

$$\Delta(A) = 3; \quad \Delta(A^{2/3}) = -9.26; \quad \Delta\left[\frac{(Z-N)^2}{4A}\right] = 0.28; \quad \Delta\left[\frac{Z^2}{A^{1/3}}\right] = 485.9.$$

The contribution from the pairing term is negligible (about 1 MeV). Using the numerical values for the coefficients in the SEMF, the energy released per fission is $E_F = 158.5$ MeV. The power of the nuclear reactor is

$$P = nE_F = 100 \text{ MW} = 6.25 \times 10^{20} \text{ MeV s}^{-1},$$

where $n$ is the number of fissions per second. Since one neutron escapes per fission and contributes to the flux, the flux $F$ is equal to the number of fissions per unit area per second, i.e.

$$F = \frac{n}{4\pi r^2} = \frac{P}{4\pi r^2 E_F} = 3.10 \times 10^{17} \text{ s}^{-1} \text{ m}^{-2}.$$

The interaction rate $R$ is given by

$$R = \sigma \times F \times (\text{number of target particles}).$$

The latter is given by $n_T = n \times N_A$, where is Avogadro's number and $n$ is found from the ideal gas law to be $n = PV/RT$, where $R$ is the ideal gas constant. Using $T = 298\ K$, $P = 1 \times 10^5$ Pa and $R = 8.31\ \mathrm{Pa\,m^3\,mol^{-1}\,K^{-1}}$, gives $n = 52.5$ mol and hence $n_T = 3.2 \times 10^{25}$. Using the cross-section $\sigma = 10^{-31}\ \mathrm{m}^2$, the rate is $9.8 \times 10^{11}\ \mathrm{s}^{-1}$.

**8.2** The neutron speed in the CM system is

$$\upsilon - m\upsilon\big/(M+m) = M\upsilon\big/(M+m)$$

and if the scattering angle in the CM system is $\theta$, then after the collision the neutron will have a speed

$$\upsilon(m + M\cos\theta)\big/(M+m)$$

in the original direction and

$$M\upsilon\sin\theta\big/(M+m)$$

perpendicular to this direction. Thus the kinetic energy is

$$E(\cos\theta) = \frac{m\upsilon^2(M^2 + 2mM\cos\theta + m^2)}{2(M+m)^2}$$

and the average value is

$$E_{final} = \bar{E} \equiv \left[\int_{-1}^{1} E(\cos\theta)\,\mathrm{d}\cos\theta\right]\left[\int_{-1}^{1}\mathrm{d}\cos\theta\right]^{-1} = RE_{initial},$$

where the reduction factor is $R = (M^2 + m^2)/(M+m)^2$. For neutron scattering from graphite, $R \approx 0.86$ and after $N$ collisions the energy will be reduced to $E_{final} = R^N E_{initial}$. The average initial energy of fission neutrons from $^{235}$U is 2 MeV and to thermalize them their energy would have to be reduced to about 0.025 eV. Thus

$$N \approx \ln(E_{final}\big/E_{initial})\big/\ln(0.86) \approx 120.$$

**8.3** From (1.57a), for the fission of $^{235}$U, $W_f = JN(235)\sigma_f$ and the total power output is $P = W_f E_f$, where $E_f$ is the energy released per fission. For the capture by $^{238}$U, $W_c = JN(238)\sigma_c$. Eliminating the flux $J$, gives

$$W_c = \frac{N(238)\sigma_c}{N(235)\sigma_f}\left(\frac{P}{E_f}\right).$$

Using the data supplied, gives $W_c = 1.08 \times 10^{19}$ atoms $\mathrm{s}^{-1} \approx 135\ \mathrm{kg\ y}^{-1}$.

**8.4** Consider fissions occurring sequentially separated by a small time interval $\delta t$. The instantaneous power is the sum of the power released from all the fissions up to that time. If $E$ is the energy released in each fission, then over the lifetime of the reactor, i.e. up to time $T$, the power is given by $P_0 = nE/T$, where $n$ is the total number of fissions and $\delta t = E/P_0$. The power after some time $t$ after the reactor has been shut down is

$$P(t) = 3(T+t)^{-1.2} + 3(T+t-\delta t)^{-1.2} + 3(T+t-2\delta t)^{-1.2}\cdots 3t^{-1.2}.$$

In this formula, the first term is the power released from the first fission and the last term is the power released from the last fission before the reactor was shut down. To sum this series, we convert it to an integral:

$$P(t) = 3 \sum_{n=0}^{n=P_0TE} (T + t - nEP_0)^{-1.2} \approx 3 \int_0^{TP_0/E} (T + t - nE/P_0)^{-1.2} \mathrm{d}n.$$

Setting $u = (T + t - nE/P_0)$, gives

$$P(t) = -3\frac{P_0}{E} \int_{T+t}^{t} u^{-1.2}\, \mathrm{d}u = 0.075 P_0 \left[t^{-0.2} - (T + t)^{-0.2}\right].$$

Using $T = 1$ yr and $t = 0.5$ yr, gives a power output of approximately 1.1 MW after six months.

**8.5** The PPI chain overall is:

$$4(^1\mathrm{H}) \to {}^4\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma + 24.68\ \mathrm{MeV}.$$

Two corrections have to be made to this. Firstly, the positrons will annihilate with electrons in the plasma releasing a further $2m_ec^2 = 1.02$ MeV per positron. Secondly, each neutrino carries off 0.26 MeV of energy into space that will not be detected. So, making these corrections, the total output per hydrogen atom is 6.55 MeV. The total energy produced to date is $5.60 \times 10^{43}$ Joules $= 3.50 \times 10^{56}$ MeV. Thus, the total number of hydrogen atoms consumed is $5.34 \times 10^{55}$ and so the fraction of the Sun's hydrogen used is 5.9% and as this corresponds to 4.6 billion years, the Sun has another 73 billion years to burn before its supply of hydrogen is exhausted.

**8.6** A solar constant of 8.4 J cm$^{-2}$ s$^{-1}$ is equivalent to $5.25 \times 10^{13}$ MeV cm$^{-2}$ s$^{-1}$ of energy deposited. If this is due to the PPI reaction

$$4(^1\mathrm{H}) \to {}^4\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma,$$

then this rate of energy deposition corresponds to a flux of $\left(5.25 \times 10^{13}/2 \times 6.55\right) \approx 4 \times 10^{12}$ neutrinos per cm$^2$ per second.

**8.7** For the Lawson criterion to be just satisfied, from (8.47),

$$L = \frac{n_d \langle \sigma_{dt} \upsilon \rangle t_c Q}{6kT} = 1.$$

We have $kT = 10$ keV and from Figure 8.7 we can estimate $\langle \sigma_{dt} \upsilon \rangle \approx 10^{-22}$ m$^3$ s$^{-1}$. Also, from (8.46), $Q = 17.6$ MeV. So, finally, $n_d = 6.8 \times 10^{18}$ m$^{-3}$.

**8.8** The mass of a *d*-*t* pair is 5.03 u $= 8.36 \times 10^{-24}$ g. The number of *d*-*t* pairs in a 1 mg pellet is therefore $1.2 \times 10^{20}$. From (8.46), each *d*-*t* pair releases 17.6 MeV of energy. Thus, allowing for the efficiency of conversion, each pellet releases $5.3 \times 10^{26}$ eV. The output power is 750 MW $= 4.7 \times 10^{27}$ eV/s. Thus the rate of pellets required is $8.9 \approx 9$ per sec.

**8.9** Assume a typical body mass of 70 kg, roughly half of which is protons. This corresponds to $2.1 \times 10^{28}$ protons and after 1 yr the number that will have decayed is $2.1 \times 10^{28}[1 - \exp(-1/\tau)]$, where $\tau$ is the lifetime of the proton in years. Each proton will eventually deposit almost all of its rest energy, i.e. approximately 0.938 GeV, in the body. Thus in 1 yr the total energy in Joules deposited per kg of body mass would be $4.5 \times 10^{16}[1 - \exp(-1/\tau)]$ and this amount will be lethal if greater than 5 Gy. Expanding the exponential gives the result that the existence of humans implies $\tau > 0.9 \times 10^{16}$ yr.

**8.10** The approximate rate of whole-body radiation absorbed is given by Equation (8.54a). Substituting the data given, we have

$$\frac{\mathrm{d}D}{\mathrm{d}t}(\mu\mathrm{Svh}^{-1}) = \frac{A(\mathrm{MBq}) \times E_\gamma(\mathrm{MeV})}{6r^2(\mathrm{m}^2)} = 1.67 \times 10^{-2}\mu\mathrm{Sv\,h}^{-1}$$

and so in 18 h, the total absorbed dose is 0.30 $\mu$Sv.

**8.11** If the initial intensity is $I_0$, then from Equation (4.17), the intensities after passing through the bone (surrounded by tissue), $I_b$, and tissue only $I_t$, are

$$I_b \approx I_0 \exp[-(\mu_b b + 2\mu_t t)] \quad \text{and} \quad I_t \approx I_0 \exp[-\mu_t(b + 2t)].$$

Thus

$$R = \exp[-b(\mu_b - \mu_t)] = 0.7$$

and hence

$$b = -\ln(0.7)/(\mu_b - \mu_t) = 2.5 \text{ cm}.$$

**8.12** From Figure 4.8, the rate of ionization energy losses is only slowly varying for momenta above about 1 GeV/c and given that living matter is mainly water and hydrocarbons a reasonable estimate is 3 MeV $\mathrm{g}^{-1}$ $\mathrm{cm}^2$. Thus the energy deposited in one year is $2.37 \times 10^9$ MeV $\mathrm{kg}^{-1}$, which is $3.8 \times 10^{-4}$ Gy.

**8.13** In general, the nuclear magnetic resonance frequency is $f = |\mu|\, B/jh$. The numerical input we use is: $j = \frac{5}{2}$, $B = 2$ T, $\mu = 3.46\ \mu_N$, giving $f = 10.8$ MHz.

## Problems B

**B.1** (a) From the definitions of $s$, $t$ and $u$, we have

$$(s + t + u)c^2 = (p_A^2 + 2p_A p_B + p_B^2) + (p_A^2 - 2p_A p_C + p_C^2) + (p_A^2 - 2p_A p_D + p_D^2),$$

which using $p_A^2 = m_A^2 c^2$ etc, becomes

$$(s + t + u)c^2 = 3m_A^2 c^2 + m_B^2 c^2 + m_C^2 c^2 + m_D^2 c^2 + 2p_A(p_B - p_C - p_D).$$

But from 4-momentum conservation, $p_A + p_B = p_C + p_D$, so that

$$(s + t + u)c^2 = 3m_A^2 c^2 + m_B^2 c^2 + m_C^2 c^2 + m_D^2 c^2 - 2p_A^2$$

and hence

$$(s+t+u) = \sum_{j=A,B,C,D} m_j^2.$$

(b) From the definition of $t$,

$$c^2 t = p_A^2 + p_C^2 - 2p_A p_C = m_A^2 c^2 + m_C^2 c^2 - 2\left(\frac{E_A E_C}{c^2} - \mathbf{p}_A \cdot \mathbf{p}_C\right).$$

For elastic scattering, $A \equiv C$. Thus $E_A = E_C$ and $|\mathbf{p}_A| = |\mathbf{p}_C| = p$, so that $\mathbf{p}_A \cdot \mathbf{p}_C = p^2 \cos\theta$. Then

$$c^2 t = 2m_A^2 c^2 - 2\left(E_A^2/c^2 - p^2\cos\theta\right)$$

and using $E_A^2 = p^2c^2 + m_A^2 c^4$, gives $t = -2p^2(1-\cos\theta)/c^2$.

**B.2** Energy conservation gives $E_\pi = E_\mu + E_\nu$, where

$$E_\pi = \gamma m_\pi c^2, \qquad E_\mu = c(m_\mu^2 c^2 + p_\mu^2)^{1/2}, \qquad E_\nu = p_\nu c$$

and hence

$$\left(\gamma m_\pi c^2 - p_\nu c\right)^2 = c^2\left(m_\mu^2 c^2 + p_\mu^2\right). \tag{1}$$

But 3-momentum conservation gives

$$p_\mu \cos\theta = p_\pi = \gamma m_\pi \upsilon, \qquad p_\mu \sin\theta = p_\upsilon = E_\nu/c. \tag{2}$$

Eliminating $p_\mu$ and $p_\nu$ between (1) and (2) and simplifying, gives

$$\tan\theta = \frac{(m_\pi^2 - m_\mu^2)}{2\beta\gamma^2 m_\pi^2}.$$

**B.3** Conservation of 4-momentum is $p_\mu = p_\pi - p_\nu$, from which $p_\mu^2 = p_\pi^2 + p_\nu^2 - 2p_\pi p_\nu$. Now $p_j^2 = m_j^2 c^2$ for $j = \pi, \mu$ and $\nu$, and

$$p_\pi p_\nu = \frac{E_\pi E_\nu}{c^2} - \mathbf{p}_\pi \cdot \mathbf{p}_\nu = m_\pi E_\nu = m_\pi |\mathbf{p}_\nu| c,$$

because $\mathbf{p}_\pi = \mathbf{0}$ and $E_\pi = m_\pi c^2$ in the rest frame of the pion. But $|\mathbf{p}_\nu| = |\mathbf{p}_\mu| \equiv p$ because the muon and neutrino emerge back-to-back. Thus, $p = (m_\pi^2 - m_\mu^2)c/2m_\pi$. But $p = \gamma m_\mu \upsilon$, from which $\upsilon = pc(p^2 + m_\mu^2 c^2)^{-1/2}$. Finally, substituting for $p$ gives

$$\upsilon = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right)c.$$

**B.4** By momentum conservation, the momentum components of $X^0$ are:

$$p_x = -0.743\,(\mathrm{GeV/c}), \quad p_y = -0.068\,(\mathrm{GeV/c}), \quad p_z = 2.595\,(\mathrm{GeV/c})$$

and hence $p_X^2 = 7.291$. Also,

$$p_A^2 = 4.686\ (\mathrm{GeV/c})^2 \quad \text{and} \quad p_B^2 = 0.304\ (\mathrm{GeV/c})^2.$$

Under hypothesis (a):

$$E_A = (m_\pi^2 c^4 + p_A^2 c^2)^{1/2} = 2.169 \text{ GeV} \quad \text{and}$$
$$E_B = (m_K^2 c^4 + p_B^2 c^2)^{1/2} = 0.740 \text{ GeV}.$$

Thus $E_X = 2.909$ GeV and $M_X = (E_X^2 - p_X^2 c^2)^{1/2} c^{-2} = 1.082 \text{ GeV/c}^2$.

Under hypothesis (b):

$$E_A = (m_p^2 c^4 + p_A^2 c^2)^{1/2} = 2.359 \text{ GeV} \quad \text{and}$$
$$E_B = (m_\pi^2 c^4 + p_B^2 c^2)^{1/2} = 0.569 \text{ GeV}.$$

Thus $E_X = 2.928$ GeV and $M_X = (E_X^2 - p_X^2 c^2)^{1/2} c^{-2} = 1.132 \text{ GeV/c}^2$.

Since $M_D = 1.86 \text{ GeV/c}^2$ and $M_\Lambda = 1.12 \text{ GeV/c}^2$, the decay is $\Lambda \to p + \pi^-$.

**B.5** If the 4-momenta of the initial and final electrons are $p = (E/c, \mathbf{q})$ and $p' = (E'/c, \mathbf{q}')$, respectively, the squared 4-momentum transfer is defined by

$$Q^2 \equiv -(p' - p)^2 = -2m^2c^2 + 2EE'/c^2 - 2\mathbf{q}\cdot\mathbf{q}'.$$

But $E = E'$ and $|\mathbf{q}| = |\mathbf{q}'| \equiv q$, so neglecting the electron mass, $Q^2 \approx 2q^2(1 - \cos\theta)$. The laboratory momentum may be found from (B.36):

$$q^2 = \frac{c^2}{4m_P^2}\left[s - (m_P - m_e)^2\right]\left[s - (m_P + m_e)^2\right] \approx \frac{c^2(s - m_P^2)^2}{4m_P^2},$$

where the invariant mass squared $s$ is defined by $s \equiv (p + P)^2/c^2$ and $P$ is the 4-momentum of the initial proton, i.e. $P = (m_P c, \mathbf{0})$. Thus,

$$s = m_e^2 + m_p^2 + 2m_p E/c^2 \approx m_P^2 + 2m_p E/c^2.$$

Substituting into the expression for $Q^2$ gives

$$Q^2 \approx 2E^2(1 - \cos\theta)/c^2.$$

**B.6** The total 4-momentum of the initial state is

$$p_{tot} = \left[\left(E + m_p c^2\right)/c, \mathbf{p}_L\right].$$

Hence the invariant mass $W$ is given by

$$(Wc^2)^2 = (E_L + m_p c^2)^2 - p_L^2 c^2,$$

where $p_L \equiv |\mathbf{p}_L|$. The invariant mass squared in the final state evaluated in the centre-of-mass frame has a minimum value $(4m_p)^2$ when all four particles are stationary. Thus, $E_{min}$ is given by

$$(E_{min} + m_p c^2)^2 - p_L^2 c^2 = (4m_p c^2)^2$$

which expanding and using

$$E_{min}^2 - p_L^2 c^2 = m_p^2 c^4,$$

gives $E_{min} = 7m_p c^2 = 6.6$ GeV.

For a bound proton, the initial 4-momentum of the projectile is $(E'_L/c, \mathbf{p}'_L)$ and that of the target is $(E/c, -\mathbf{p})$, where $\mathbf{p}$ is the internal momentum of the nucleons, which we have taken to be in the opposite direction of the beam because this gives the maximum invariant mass for a given $E'_L$. The invariant mass $W'$ is now given by

$$(W'c^2)^2 = (E'_L + E)^2 - (p'_L - p)^2c^2 = 2m_p^2c^4 + 2EE'_L + 2pp'_Lc^2.$$

Since the thresholds $E_{min}$ and $E'_{min}$ correspond to the same invariant mass $4m_p$, we have

$$2m_pc^2E_{min} = 2EE'_{min} + 2pp'_{min}c^2.$$

Finally, since the internal momentum of the nucleons is $\sim 250$ MeV/c (see Chapter 7), $E \approx m_pc^2$, while for the relativistic incident protons $p'_{min} \approx E'_{min}/c$, so using these gives

$$E'_{min} \approx \left(1 - p/m_pc\right)E_{min} = 4.8 \text{ GeV}.$$

**B.7** The initial total energy is $E_i = E_A = m_Ac^2$ and the final total energy is $E_f = E_B + E_C$, where

$$E_B = (m_B^2c^4 + p_B^2c^2)^{1/2}, \quad \text{and} \quad E_C = (m_C^2c^4 + p_C^2c^2)^{1/2},$$

with $p_B = |\mathbf{p}_B|$ and $p_C = |\mathbf{p}_C|$. But by momentum conservation, $\mathbf{p}_B = -\mathbf{p}_C \equiv \mathbf{p}$ and so

$$\left[m_Ac^2 - (m_B^2c^4 + p^2c^2)^{1/2}\right]^2 = \left(m_C^2c^4 + p^2c^2\right),$$

which on expanding gives $E_B = \left(m_A^2 + m_B^2 - m_C^2\right)c^2/2m_A$.

**B.8** If the 4-momenta of the photons are $p_i = (E_i/c, \mathbf{p}_i)(i = 1, 2)$, then the invariant mass of $M$ is given by

$$M^2c^4 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)c^2 = 2E_1E_2(1 - \cos\theta),$$

since

$\mathbf{p}_1 \cdot \mathbf{p}_2 = E_1E_2(1 - \cos\theta)/c^2$ for zero-mass photons. Thus,

$$\cos\theta = 1 - M^2c^4/2E_1E_2.$$

**B.9** A particle with velocity $\upsilon$ will take time $t = L/\upsilon$ to pass between the two counters. Relativistically, $p = m\upsilon\gamma$ with $\gamma = (1 - \upsilon^2/c^2)^{-1/2}$. Solving, gives $\upsilon = c(1 + m^2c^2/p^2)^{-1/2}$ and hence the difference in times-of-flight (assuming $m_1 > m_2$) is

$$\Delta t = \frac{L}{c}\left[\left(1 + \frac{m_1^2c^2}{p^2}\right)^{1/2} - \left(1 + \frac{m_2^2c^2}{p^2}\right)^{1/2}\right]$$

Using

$$m_1c^2 = m_pc^2 = 0.938 \text{ GeV}, \quad m_2c^2 = m_\pi c^2 = 0.140 \text{ GeV} \quad \text{and} \quad pc = 2 \text{ GeV},$$

gives $\Delta t = (1.105 - 1.002)(L/c)$ and $L_{min} = 0.58$ m.

**B.10** In an obvious notation, the kinematics in the lab frame are:

$$\gamma(E_\gamma, \mathbf{p}_\gamma) + e^-(mc^2, 0) \to \gamma(E'_\gamma, \mathbf{p}'{}_\gamma) + e^-(E, \mathbf{p}).$$

Energy conservation gives $E_\gamma + mc^2 = E'_\gamma + E$ and momentum conservation gives $\mathbf{p}_\gamma = p'_\gamma + \mathbf{p}_e$. From the latter we have

$$E_e^2 - m^2c^4 = c^2(\mathbf{p}_\gamma^2 + \mathbf{p}'^2_\gamma - 2\mathbf{p}_\gamma.\mathbf{p}'_\gamma).$$

But $p_\gamma c = E_\gamma$, $p'_\gamma c = E'_\gamma$ and the scattering angle is $\theta$, so we have

$$E_e^2 - m^2c^4 = E_\gamma^2 + E'^2_\gamma - 2E_\gamma E'_\gamma \cos\theta.$$

Eliminating $E$ between this equation and the equation for energy conservation gives

$$E'_\gamma = E_\gamma[1 + E_\gamma(1 - \cos\theta)\big/mc^2]^{-1}.$$

Finally, using $E'_\gamma = E_\gamma/2$ and $\theta = 60^0$, gives $E_\gamma = 2mc^2 = 1.02$ MeV.

## Problems C

**C.1** The assumptions are: ignore recoil of target nucleus because its mass is much greater than the total energy of the projectile $\alpha$ particle; use nonrelativistic kinematics because the kinetic energy of the $\alpha$ particle is very much less than its rest mass; assume Rutherford formula (i.e. the Born approximation) is valid for small-angle scattering. The relevant formula is then (C.13) and it may be evaluated using $z = 2$, $Z = 83$, $E_{kin} = 20$ MeV and $\theta = 20^0$. The result is $\mathrm{d}\sigma/\mathrm{d}\Omega = 98.3$ b/sr.

**C.2** From Figure C.2, the distance of closest approach $d$ is when $x = 0$. For $x < 0$, the sum of the kinetic and potential energies is $E_{ke} = \frac{1}{2}m\upsilon^2$ and the angular momentum is $m\upsilon b$. At $x = 0$, the total mechanical energy is

$$\frac{mu^2}{2} + \frac{Zze^2}{4\pi\varepsilon_0 d}$$

and the angular momentum is $mud$, where $u$ is the instantaneous velocity. From angular momentum conservation, $u = \upsilon b/d$ and using this in the conservation of total mechanical energy gives $d^2 - Kd - b^2 = 0$, where, using (C.9), $K \equiv 2b/\cot(\theta 2)$. The solution for $d \geq 0$ is

$$d = b\left[1 + \mathrm{cosec}(\theta/2)\right]\big/\cot(\theta/2).$$

**C.3** The result for small-angle scattering follows directly from (C.9) in the limit $\theta \to 0$. Evaluating $b$, we have, using the data given,

$$b = \frac{zZe^2}{2\pi\varepsilon_0 m\upsilon^2\theta} = 2zZ\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)\frac{\hbar c}{mc^2}\frac{1}{(\upsilon/c)^2\theta} = 1.55 \times 10^{-13}\ \mathrm{m}.$$

The cross-section for scattering through an angle greater than $5^0$ is thus $\sigma = \pi b^2 = 7.55 \times 10^{-26}$ m$^2$ and the probability that the proton scatters through an angle greater than $5^0$ is $P = 1 - \exp(-n\sigma t)$, where $n$ is the number density of the target. Using $n = 6.658 \times 10^{28}$ m$^{-3}$, gives $P = 4.91 \times 10^{-2}$. Since $P$ is very small but the number of scattering centres is very large, the scattering is governed by the Poisson distribution and the probability for a single scatter is

$$P_1(m) = me^{-m} = 4.91 \times 10^{-2}, \text{ giving } m \approx 0.052.$$

Finally, the probability for two scattering is

$$P_2 = m^2 \exp(-m)\big/2! \approx 1.3 \times 10^{-3}.$$